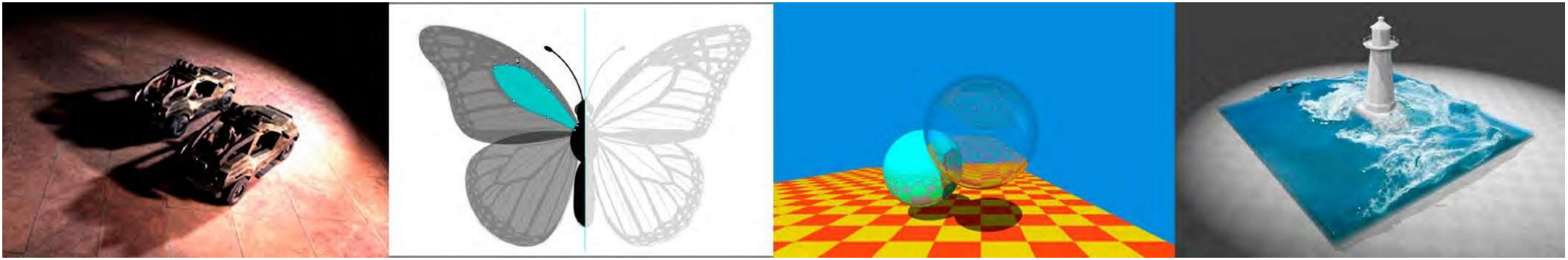
Computer Graphics

Geometry 2 (Curves and Surfaces)



Last Lecture

- Introduction to geometry
  - Examples of geometry
  - Various representations of geometry
    - Implicit
    - Explicit

Today

- Explicit Representations
- Curves
  - Bezier curves
  - De Casteljau's algorithm
  - B-splines, etc.
- Surfaces
  - Bezier surfaces
  - Triangles & quads
    - Subdivision, simplification, regularization

Explicit Representations in Computer Graphics

Many Explicit Representations in Graphics

triangle meshes

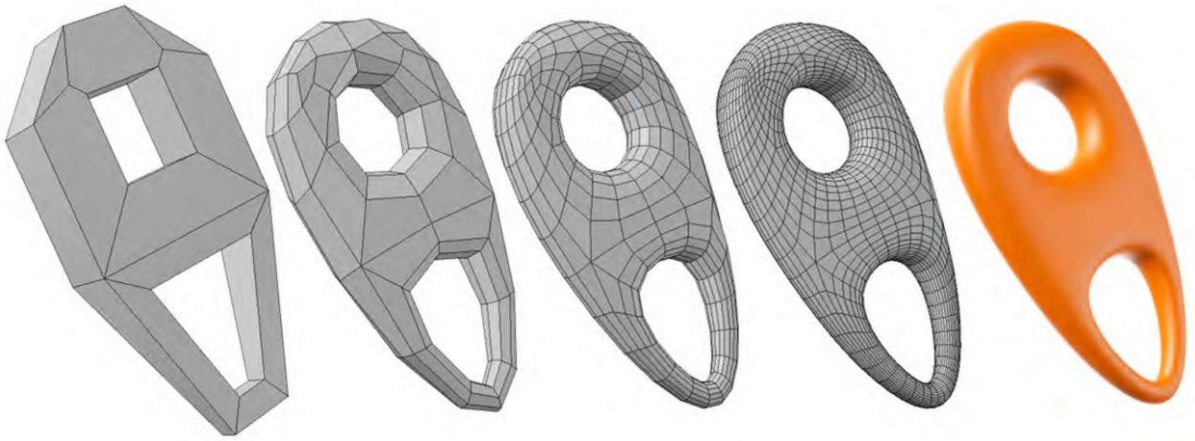
Bezier surfaces

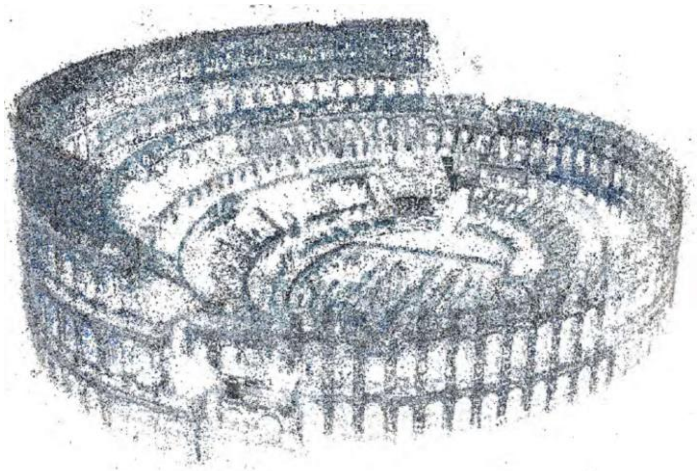
subdivision surfaces

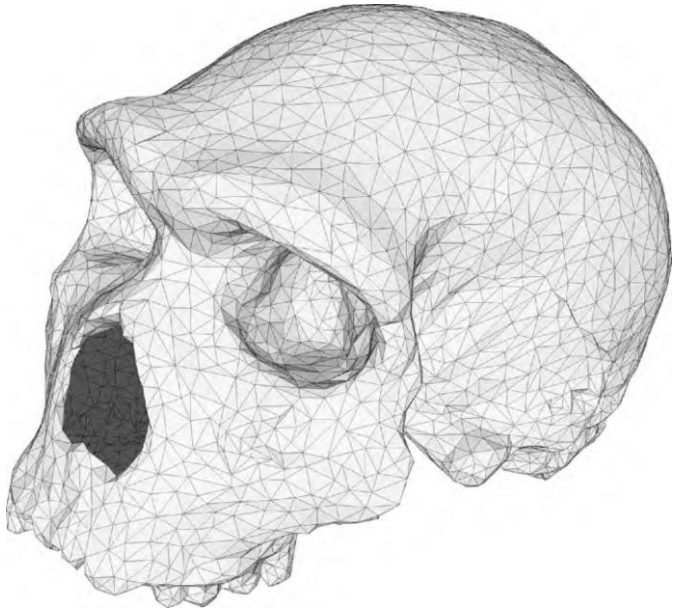
NURBS

point clouds

...

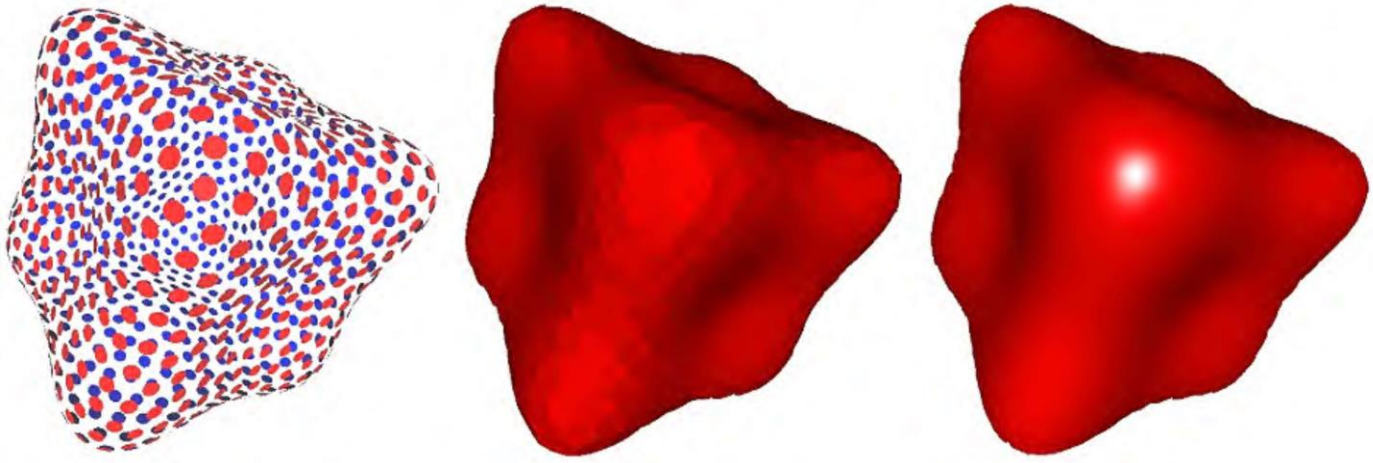






Point Cloud (Explicit)

- Easiest representation: list of points (x,y,z)
- Easily represent any kind of geometry
- Useful for LARGE datasets (>>1 point/pixel)
- Often converted into polygon mesh
- Difficult to draw in undersampled regions



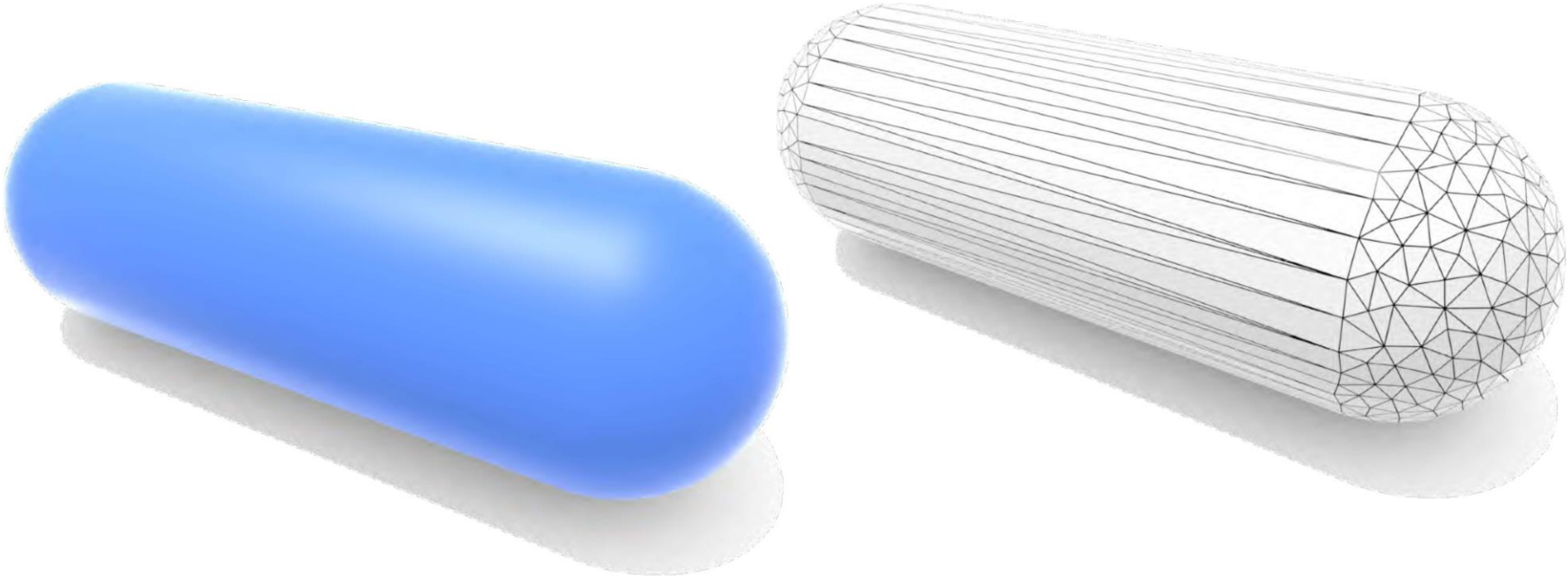
Polygon Mesh (Explicit)

Store vertices & polygons (often triangles or quads)

Easier to do processing / simulation, adaptive sampling

More complicated data structures

Perhaps most common representation in graphics



The Wavefront Object File (.obj) Format

Commonly used in Graphics research

Just a text file that specifies vertices, normals, texture coordinates and their connectivities

```
# This is a comment

v 1.000000 -1.000000 -1.000000
v 1.000000 -1.000000 1.000000
v -1.000000 -1.000000 1.000000
v -1.000000 -1.000000 -1.000000
v 1.000000 1.000000 -1.000000
v 0.999999 1.000000 1.000001
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vt 0.748573 0.750412
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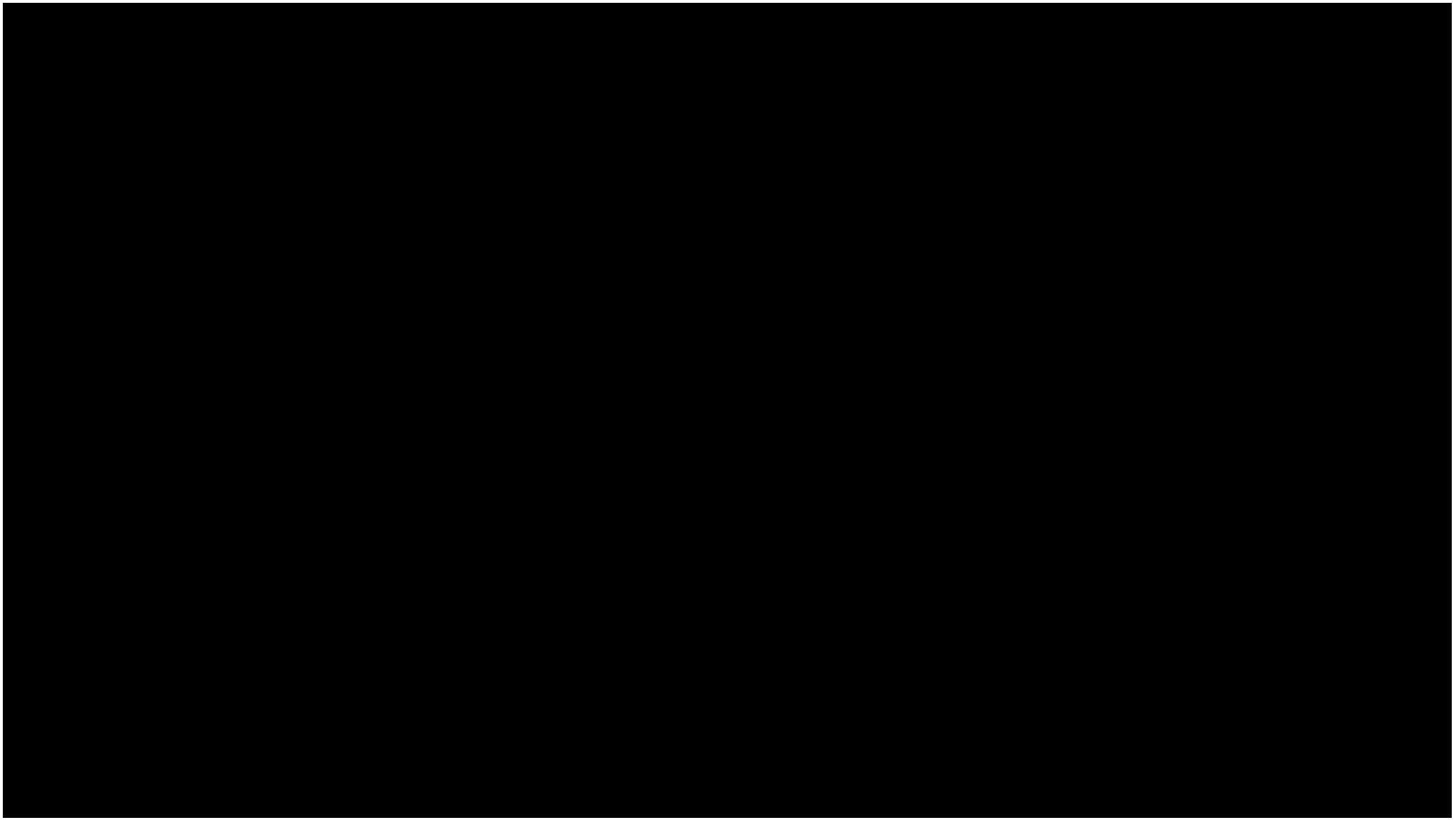
f 8/11/7 7/12/7 6/10/7
f 1/2/8 2/9/8 3/13/8
f 1/2/8 3/13/8 4/14/8
```

https://www.meshlab.net/

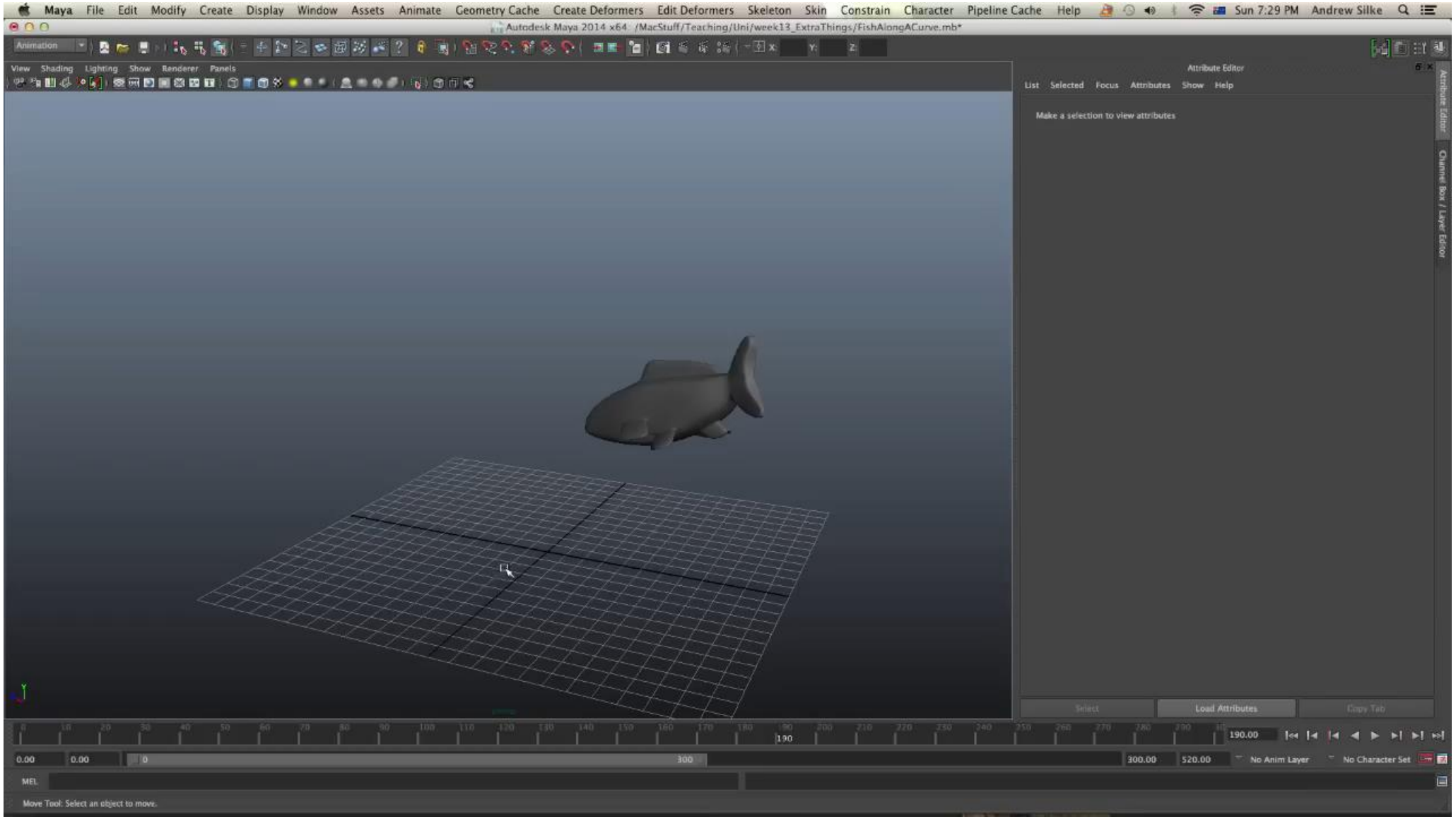
Curves

Camera Paths



Flythrough of proposed Perth Citylink subway, https://youtu.be/rIJMuQPwr3E

Animation Curves

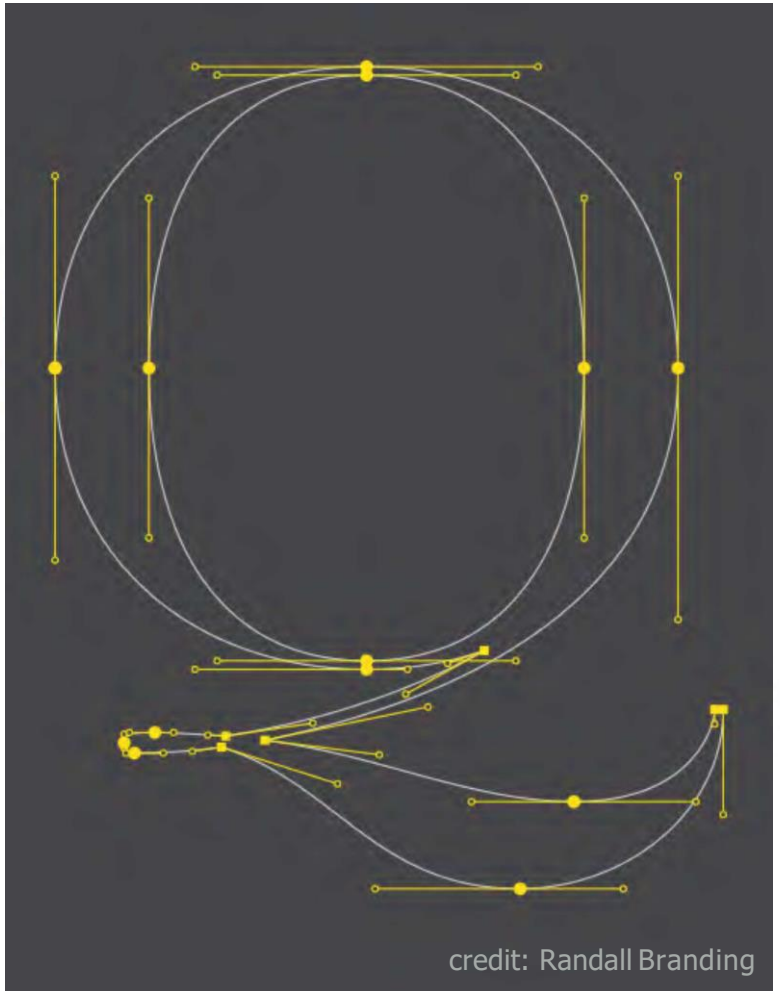


Maya Animation Tutorial: https://youtu.be/b-o5wtZlJPc

Vector Fonts

The Quick Brown
Fox Jumps Over
The Lazy Dog

ABCDEFGHIJKLMNOPQRSTUVWXYZ
abcdefghijklmnopqrstuvwxyz 0123456789

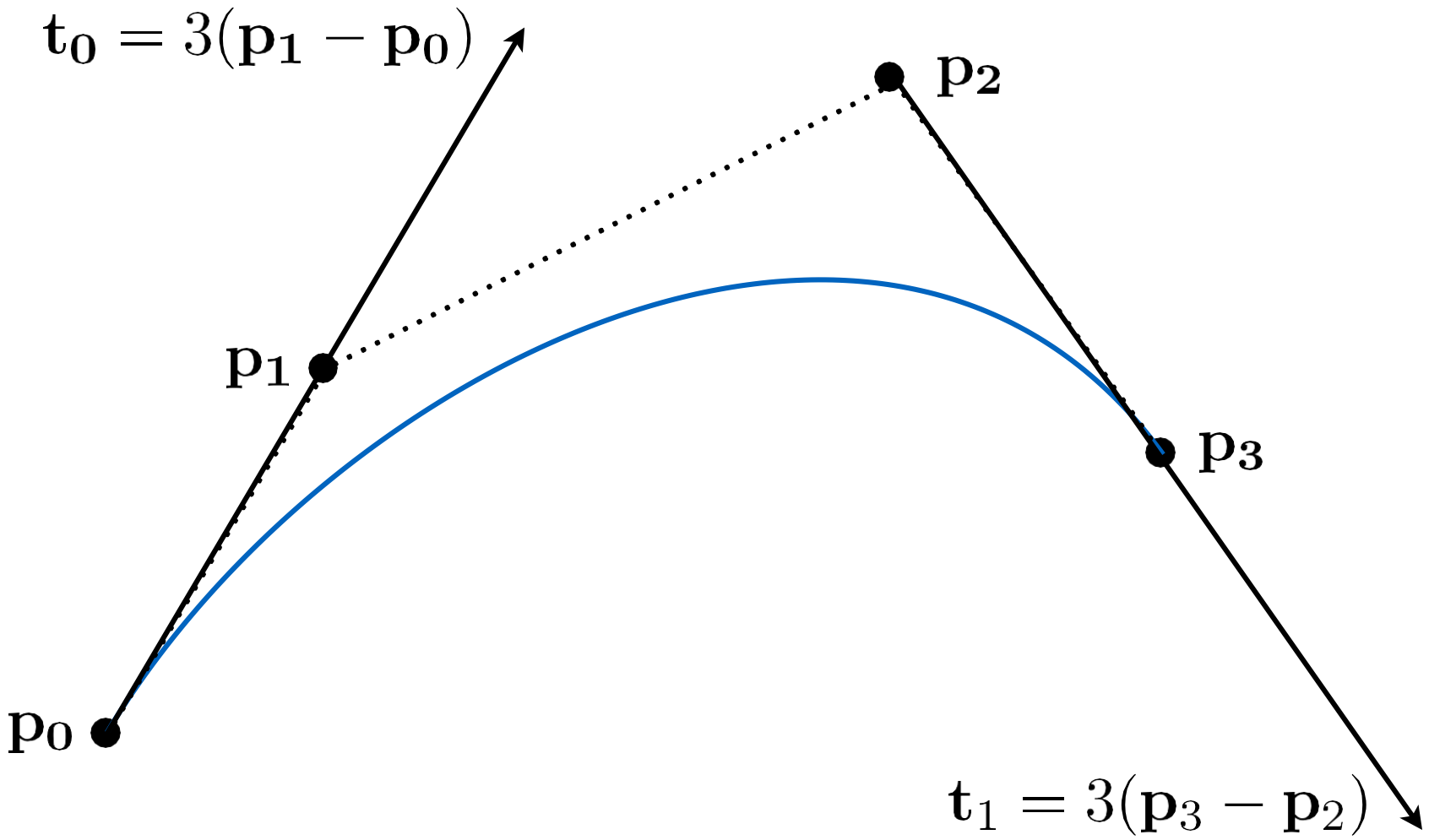


Baskerville font - represented as piecewise cubic Bézier curves

Bézier Curves

(贝塞尔曲线)

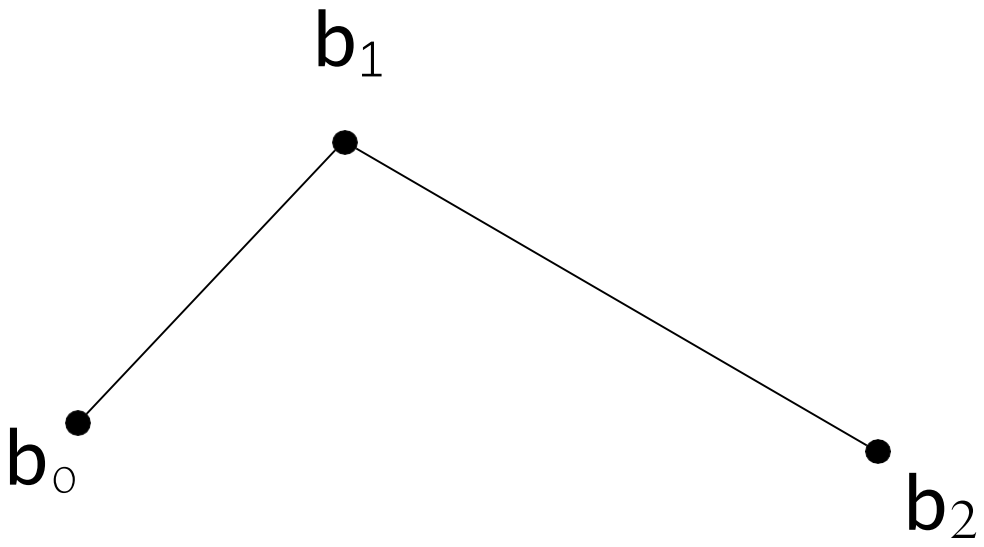
Defining Cubic Bézier Curve With Tangents



Evaluating Bézier Curves (de Casteljau Algorithm)

Bézier Curves – de Casteljau Algorithm

Consider three points (quadratic Bezier)

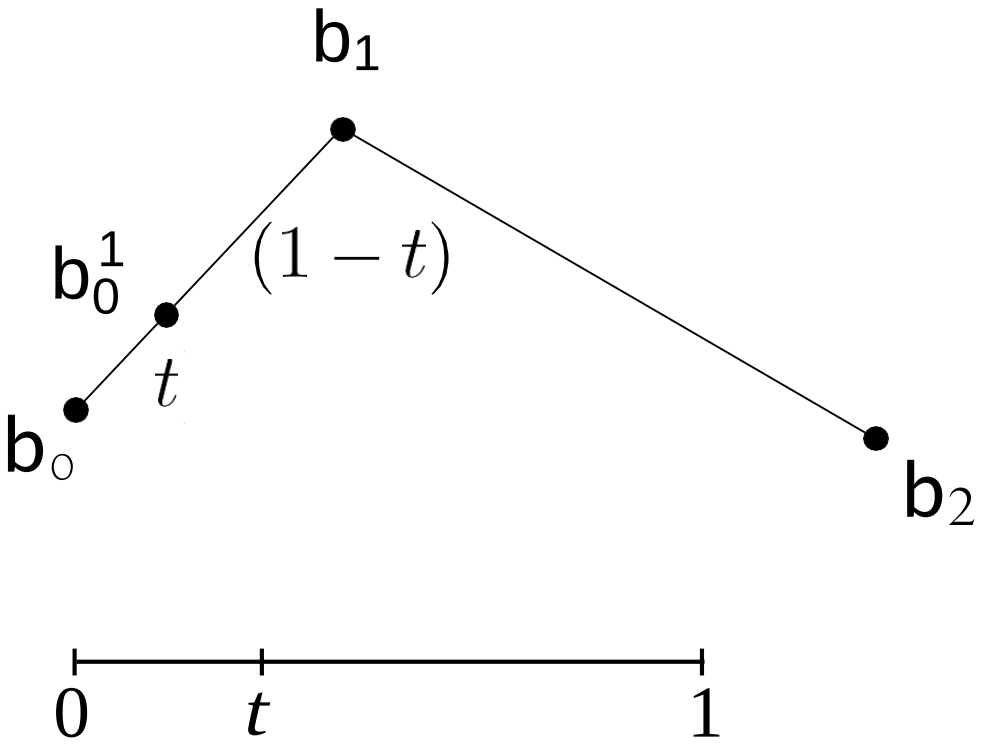


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert a point using linear interpolation



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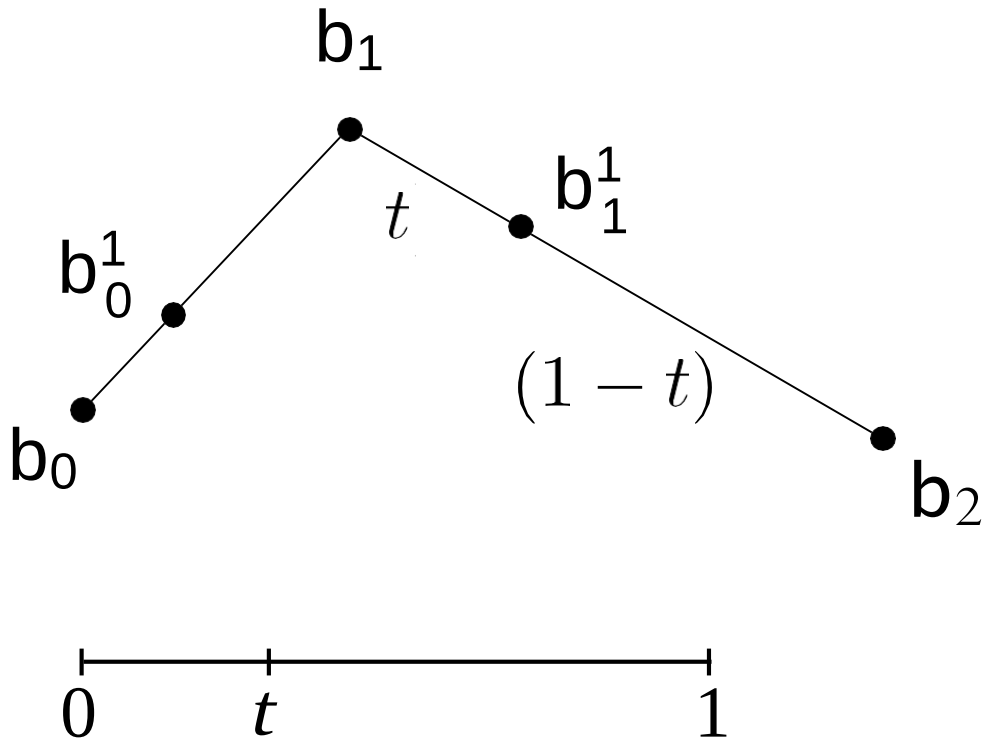
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Insert on both edges

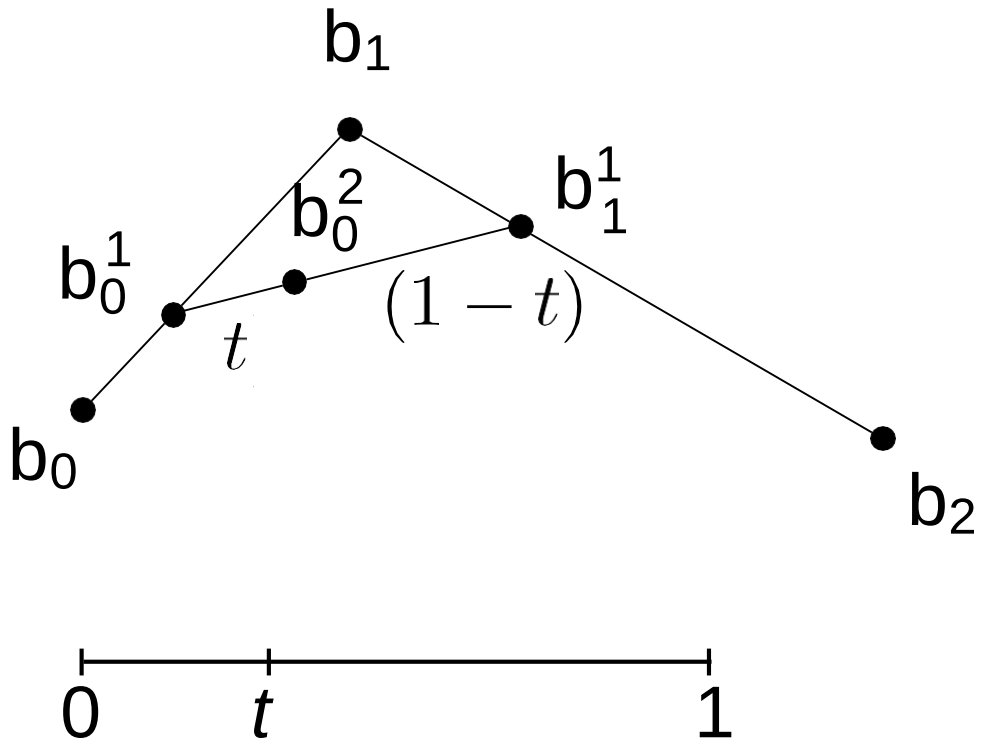


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Repeat recursively

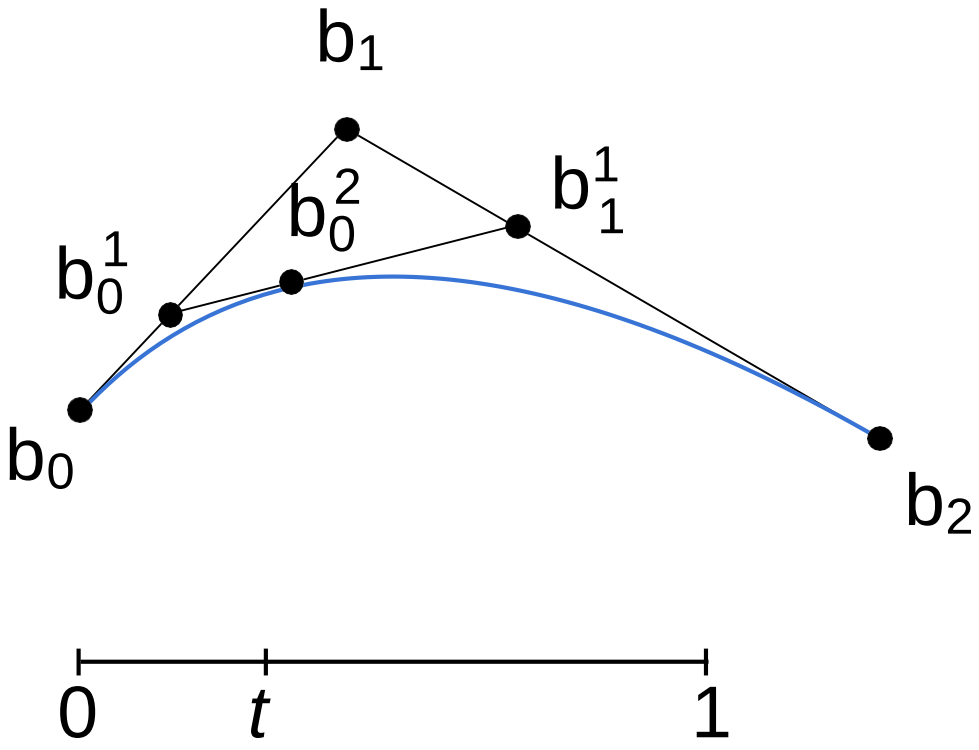


Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

Bézier Curves – de Casteljau Algorithm

Run the same algorithm for every t in [0,1]



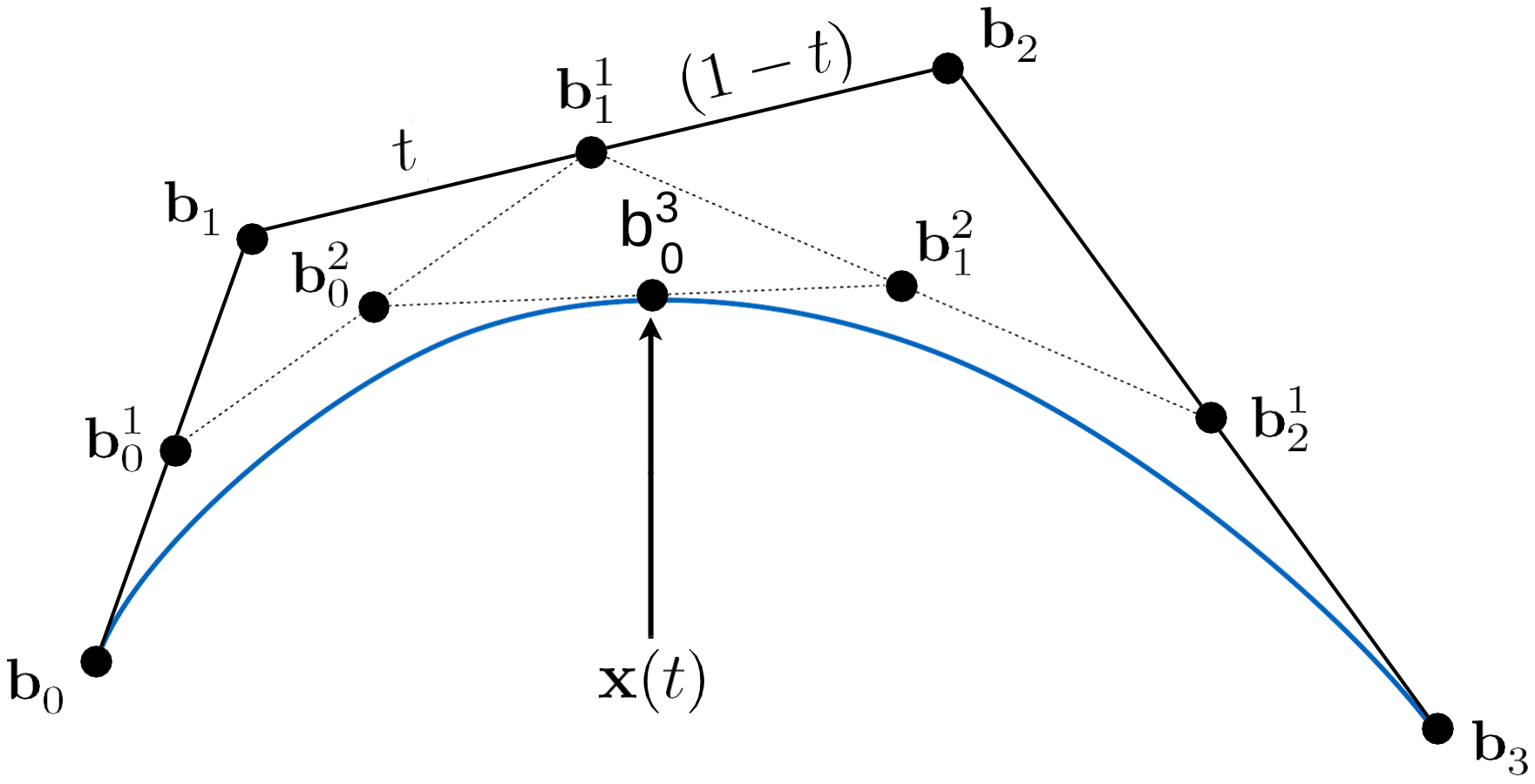
Pierre Bézier
1910 – 1999

Paul de Casteljau
b. 1930

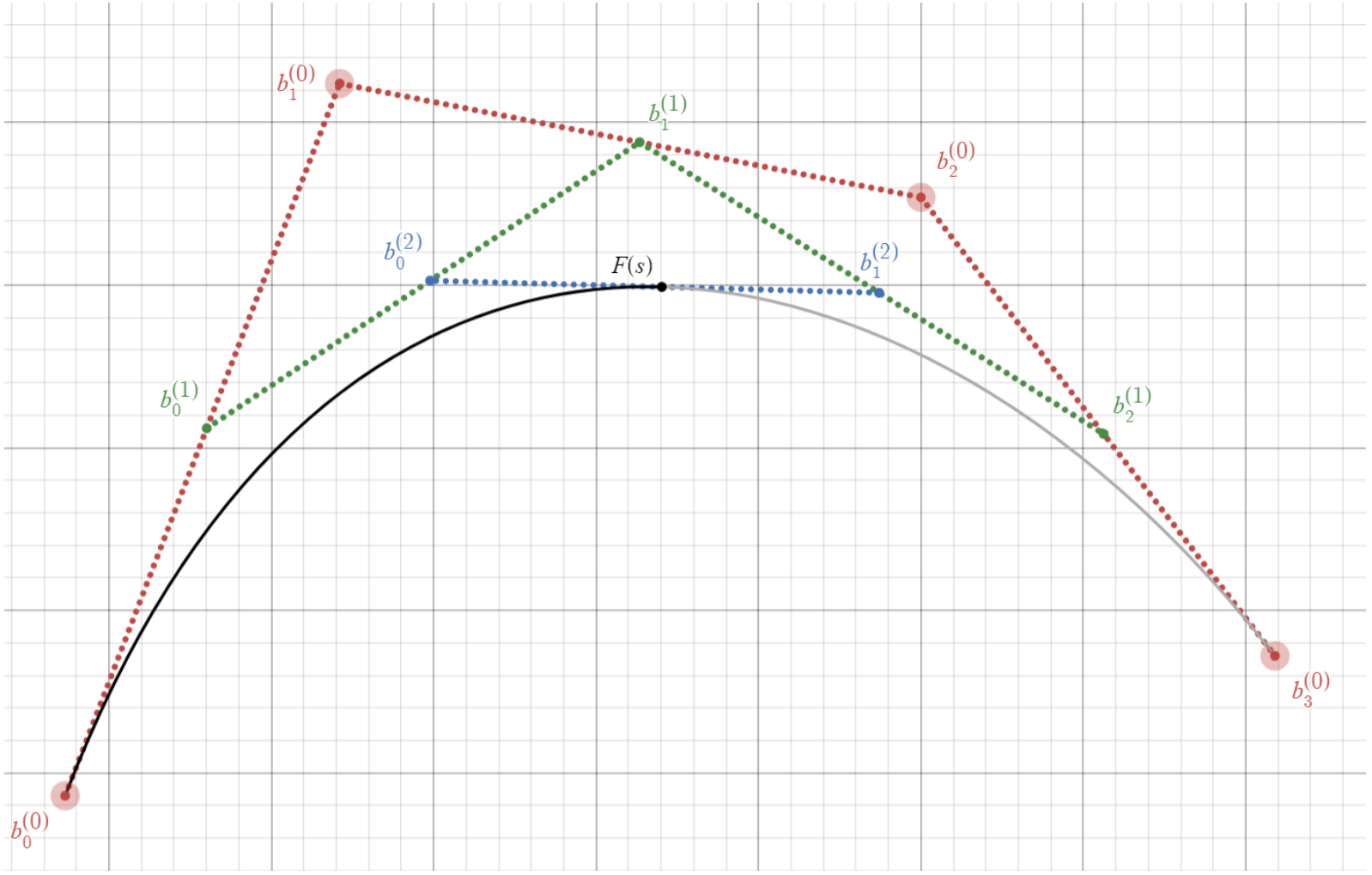
Cubic Bézier Curve – de Casteljau

Four input points in total

Same recursive linear interpolations



Visualizing de Casteljau Algorithm

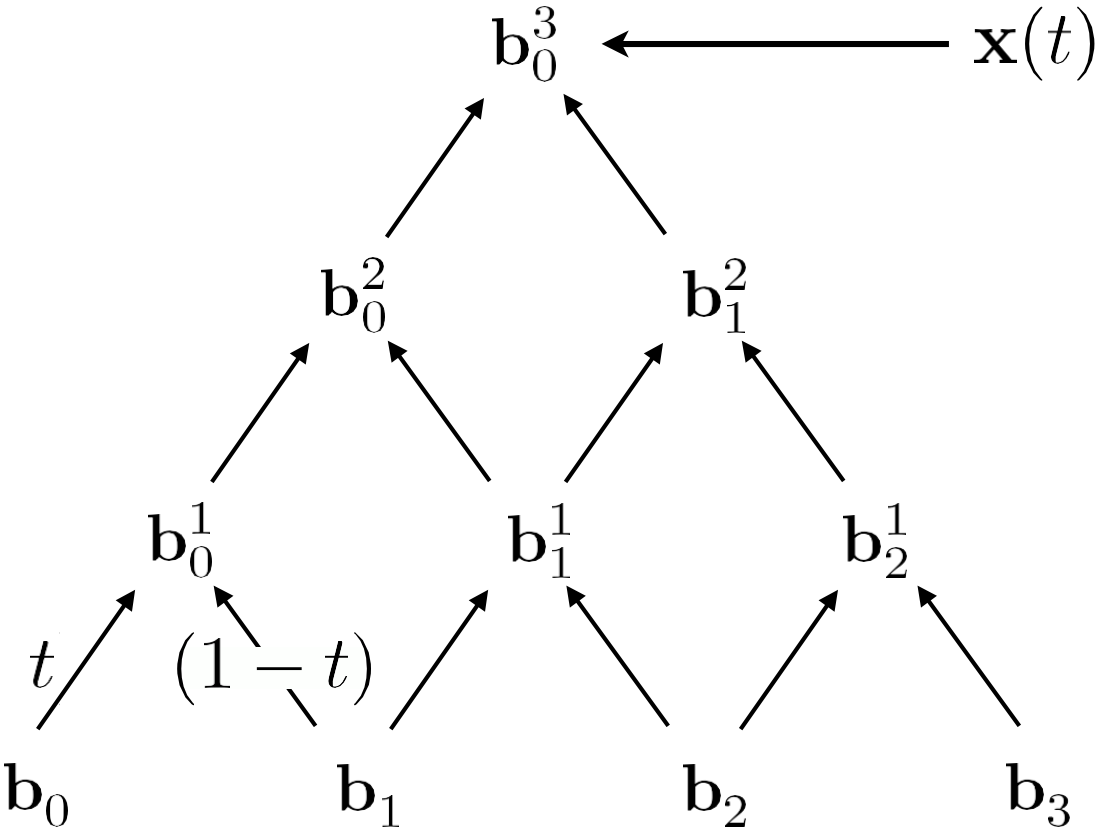


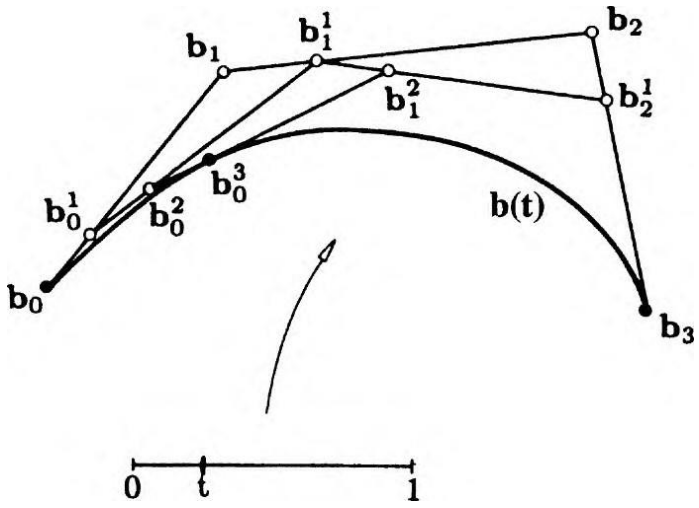
https://www.desmos.com/calculator/s78usaowv9

Evaluating Bézier Curves
Algebraic Formula

Bézier Curve – Algebraic Formula

de Casteljau algorithm gives a pyramid of coefficients

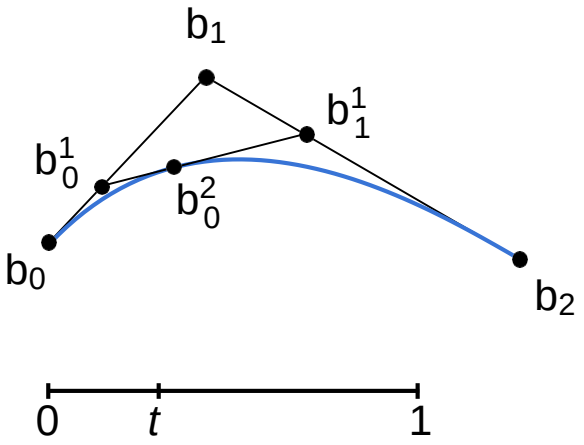




Every rightward arrow is multiplication by t,
Every leftward arrow by (1–t)

Bézier Curve – Algebraic Formula

Example: quadratic Bézier curve from three points



$$b_0^1(t) = (1 - t)b_0 + tb_1$$

$$b_1^1(t) = (1 - t)b_1 + tb_2$$

$$b^2(t) = (1 - t)b_0^1 + tb_1^1$$

$$\mathbf{b}_0^2(t) = (1-t)^2\mathbf{b}_0 + 2t(1-t)\mathbf{b}_1 + t^2\mathbf{b}_2$$

Bézier Curve – General Algebraic Formula

Bernstein form of a Bézier curve of order n:

$$\mathrm{b}^n(t) = \mathrm{b}_0^n(t) = \sum_{j=0}^{n} \mathrm{b}_j B_j^n(t)$$

Bézier curve order n

(vector polynomial of degree n)

Bernstein polynomial

(scalar polynomial of degree n)

Bézier control points

(vector in R^N)

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Last Lecture

- Perspective Projection
- Introduction to geometry
  - Implicit Representation of Geometry
  - Explicit Representation of Geometry
- Curves
  - Bezier curves
  - De Casteljau's algorithm （德卡斯特里奥算法）
  - B-splines, etc.

Perspective Projection

- How to do perspective projection
  - First "squish" the frustum into a cuboid (n -> n, f -> f) ($M_{persp->ortho}$)
  - Do orthographic projection (M_{ortho}, already known!)

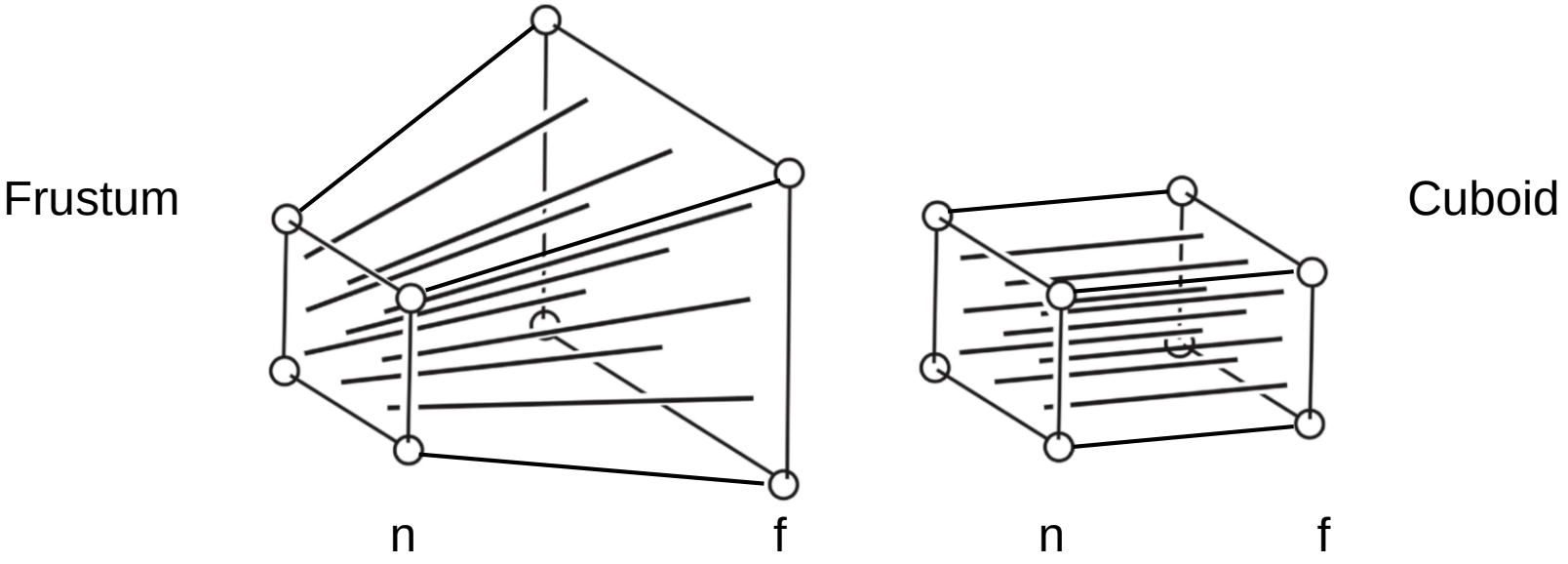


Fig. 7.13 from *Fundamentals of Computer Graphics, 4th Edition*

Perspective Projection

$$M_{ortho} = \begin{bmatrix} \frac{2}{r-l} & 0 & 0 & 0 \\ 0 & \frac{2}{t-b} & 0 & 0 \\ 0 & 0 & \frac{2}{n-f} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -\frac{r+l}{2} \\ 0 & 1 & 0 & -\frac{t+b}{2} \\ 0 & 0 & 1 & -\frac{n+f}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$M_{persp \to ortho} = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n+f & -nf \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_{persp} = M_{ortho} M_{persp \to ortho}$$

Many Implicit Representations in Graphics

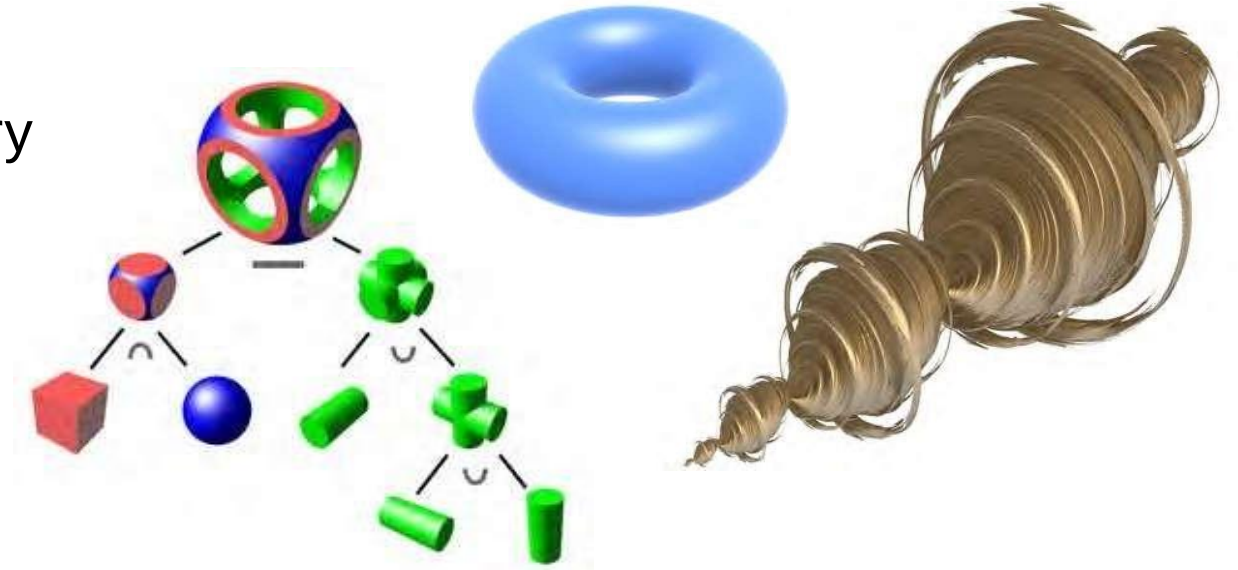
Algebraic surfaces

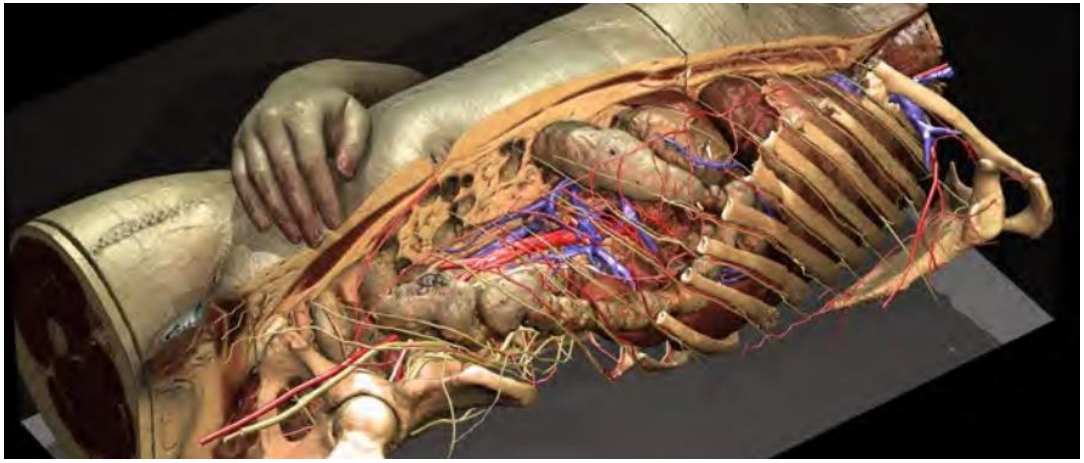
Constructive solid geometry

Level set methods

Fractals （分形）

...





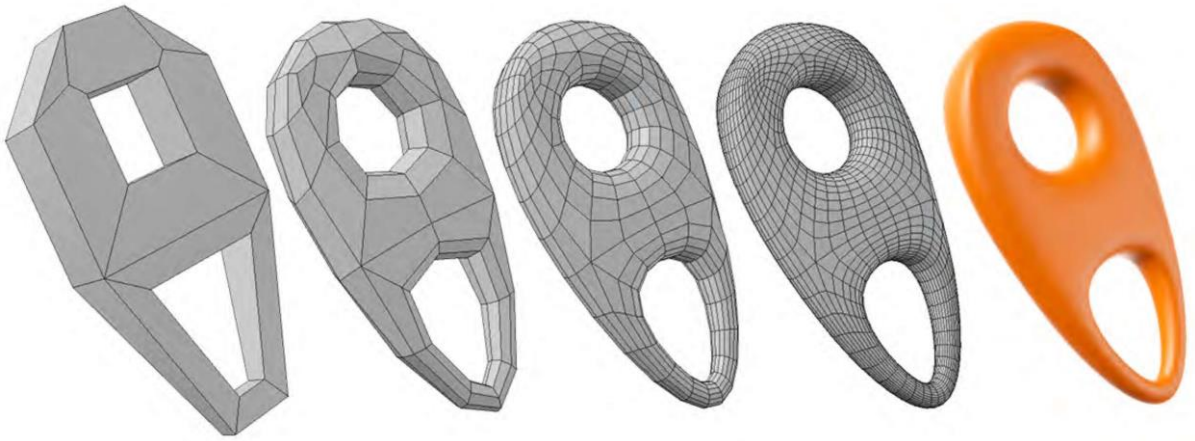
Many Explicit Representations in Graphics

triangle meshes

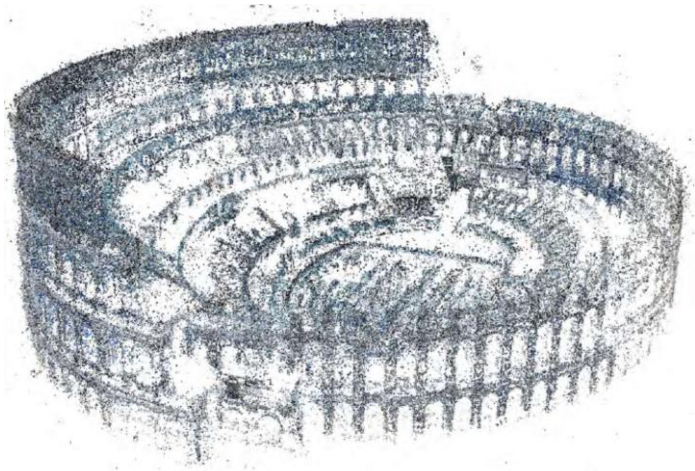
Bezier surfaces

subdivision surfaces

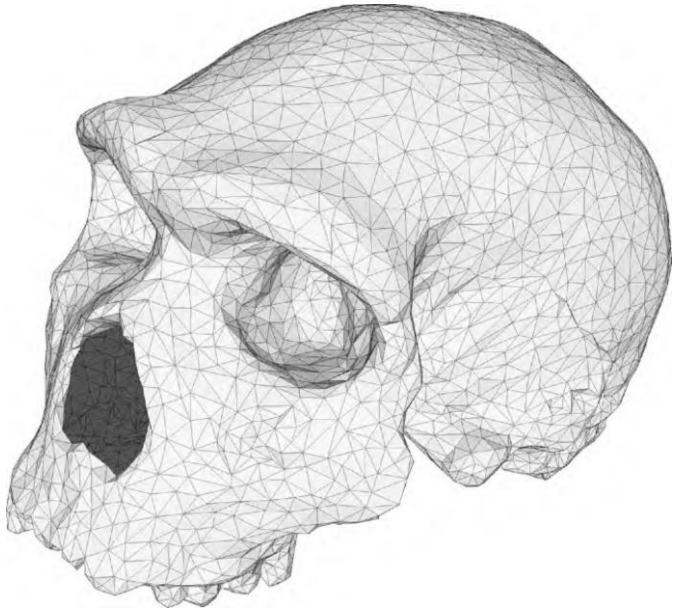
NURBS



point clouds



...



The Wavefront Object File (.obj) Format

Commonly used in Graphics research

Just a text file that specifies vertices, normals, texture coordinates and their connectivities

```
# This is a comment

v 1.000000 -1.000000 -1.000000
v 1.000000 -1.000000 1.000000
v -1.000000 -1.000000 1.000000
v -1.000000 -1.000000 -1.000000
v 1.000000 1.000000 -1.000000
v 0.999999 1.000000 1.000001
v -1.000000 1.000000 1.000000
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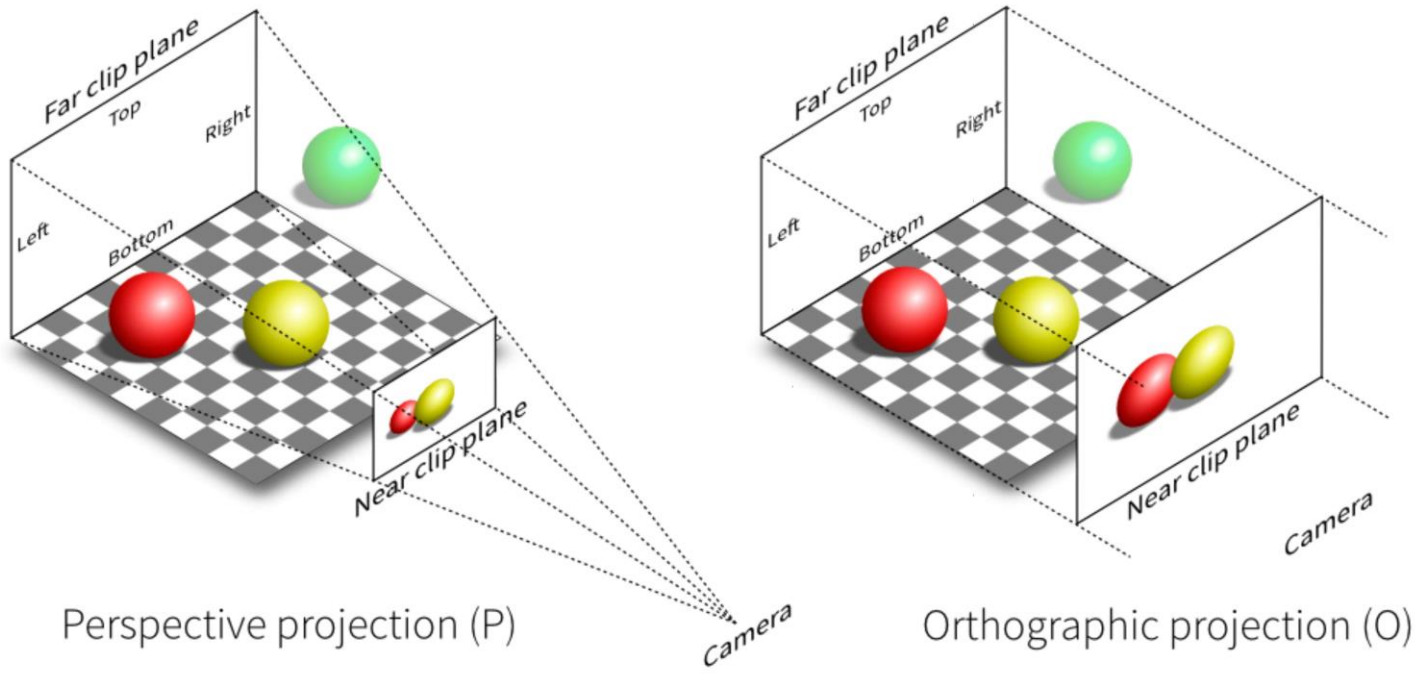
f 1/2/8 2/9/8 3/13/8
f 1/2/8 3/13/8 4/14/8
```

https://www.meshlab.net/

主观题 10分

设置

为什么视锥要设置f和n?



作答

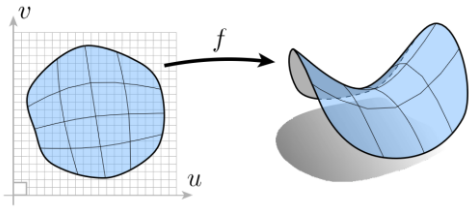
多选题 1分

设置

请问下面哪些表达属于几何隐式表达?

A $x^2+y^2+z^2 = 1$ （ algebraic surface ）

B $f : \mathbb{R}^2 \to \mathbb{R}^3; (u, v) \mapsto (x, y, z)$



C 基于布尔操作的CSG

D 水平集函数（Level Set Methods）

E Bezier Curve (贝塞尔曲线)

提交

Half-edge data structure

```
# This is a comment

v 1.000000 -1.000000 -1.000000
v 1.000000 -1.000000 1.000000
v -1.000000 -1.000000 1.000000
v -1.000000 -1.000000 -1.000000
v 1.000000 1.000000 -1.000000
v 0.999999 1.000000 1.000001
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f 2/9/3 6/10/3 3/5/3
f 6/10/4 7/6/4 3/5/4
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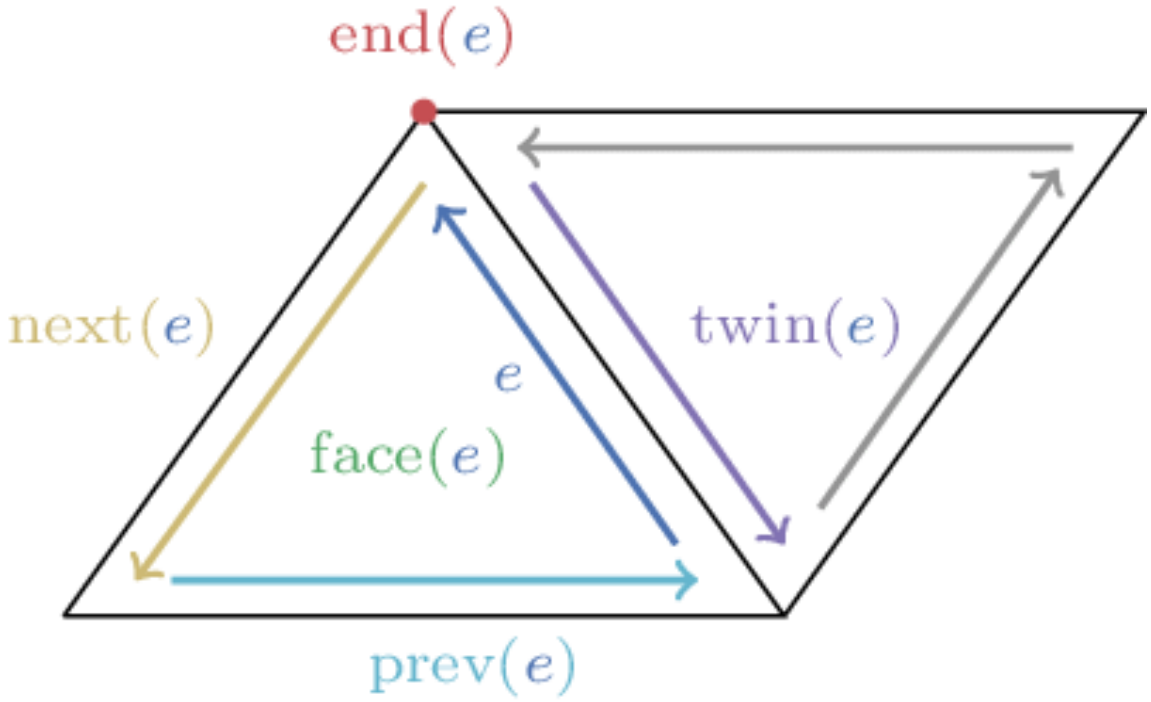
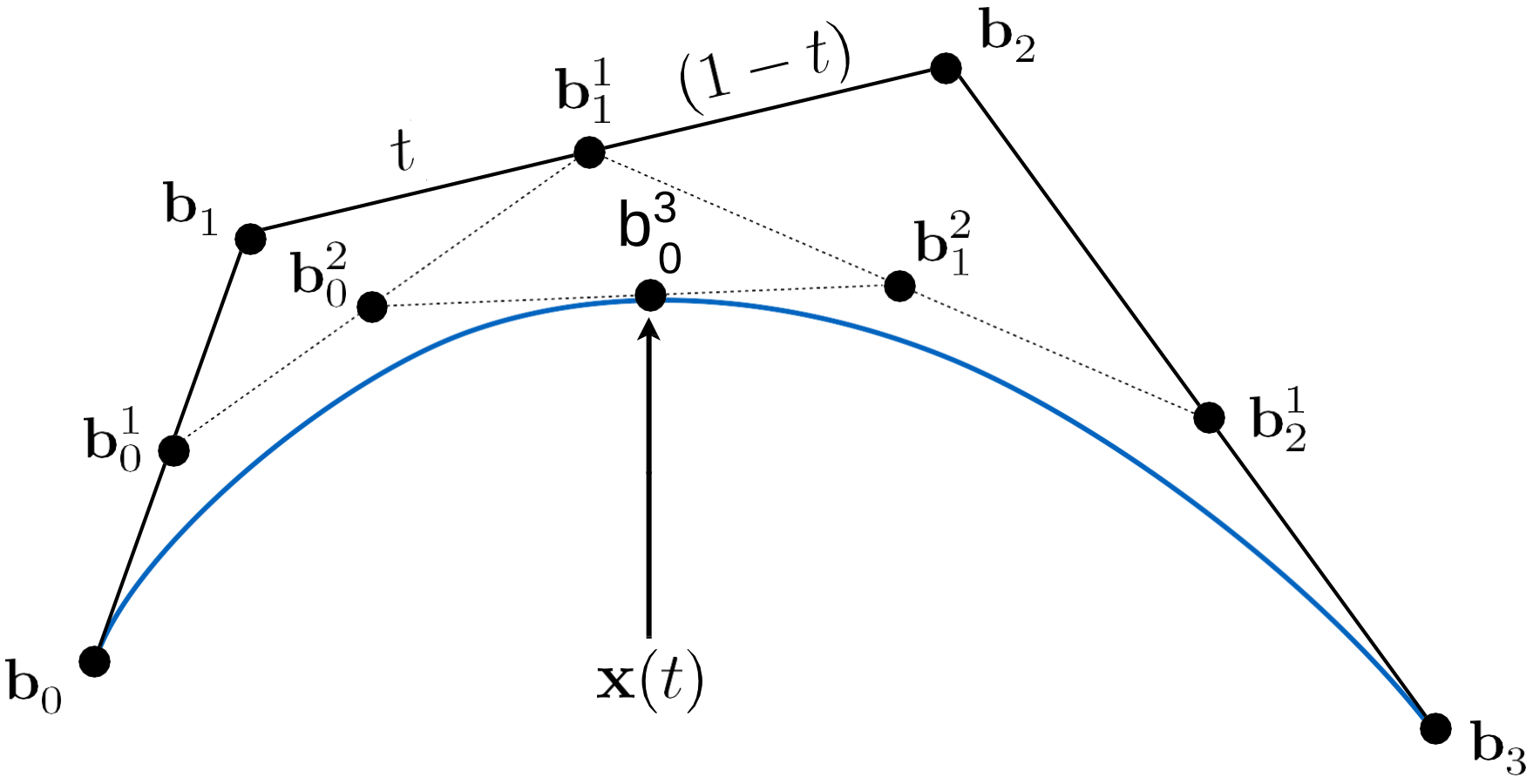


https://jerryyin.info/geometry-processing-algorithms/half-edge/

Cubic Bézier Curve – de Casteljau

Four input points in total

Same recursive linear interpolations



Bézier Curve – General Algebraic Formula

Bernstein form of a Bézier curve of order n:

$$\mathrm{b}^n(t) = \mathrm{b}_0^n(t) = \sum_{j=0}^{n} \mathrm{b}_j B_j^n(t)$$

Bézier curve order n

(vector polynomial of degree n)

Bernstein polynomial

(scalar polynomial of degree n)

Bézier control points

(vector in R^N)

Bernstein polynomials:

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Bézier Curve – Algebraic Formula: Example

Bernstein form of a Bézier curve of order n:

$$\mathbf{b}^n(t) = \sum_{j=0}^{n} \mathbf{b}_j B_j^n(t)$$

Example: assume n = 3 and we are in R^3

i.e. we could have control points in 3D such as:

$$\mathbf{b}_0 = (0, 2, 3),\ \mathbf{b}_1 = (2, 3, 5),\ \mathbf{b}_2 = (6, 7, 9),\ \mathbf{b}_3 = (3, 4, 5)$$

These points define a Bezier curve in 3D that is a cubic polynomial in t:

$$\mathbf{b}^n(t) = \mathbf{b}_0 (1-t)^3 + \mathbf{b}_1 3t(1-t)^2 + \mathbf{b}_2 3t^2(1-t) + \mathbf{b}_3 t^3$$

Properties of Bernstein polynomial

- 非负 (Non-negative)

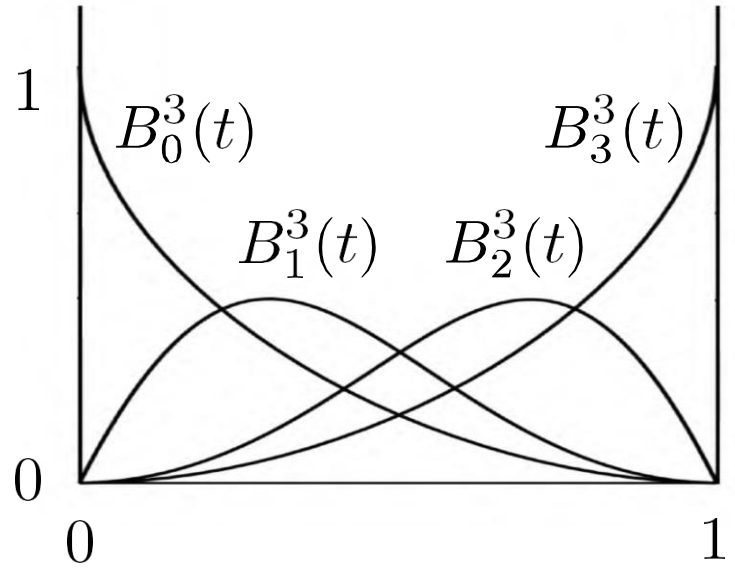
$$B_{i,n}(t) = \begin{cases} = 0 & t = 0,1 \\ > 0 & t \in (0,1), i = 1,2,\cdots,n-1; \end{cases}$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

- 端点(End point)

$$B_{i,n}(0) = \begin{cases} 1 & (i = 0) \\ 0 & otherswise \end{cases}$$

$$B_{i,n}(1) = \begin{cases} 1 & (i = n) \\ 0 & otherswise \end{cases}$$



Properties of Bernstein polynomial

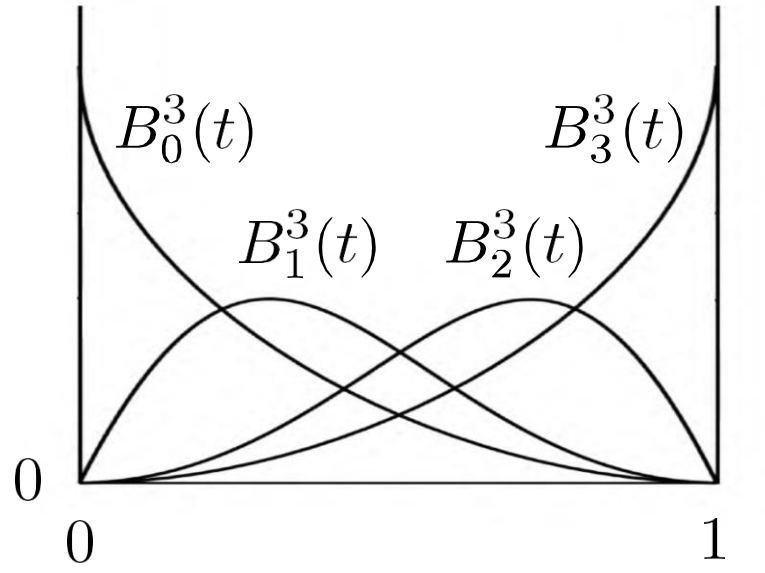
- 归一性(Unity)

$$\sum_{i=0}^{n} B_{i,n}(t) \equiv 1 \qquad t \in (0,1)$$

  - 证明: 根据二项式定理(*Binomial Theorem*), 可以得到

$$\sum_{i=0}^{n} B_{i,n}(t) = \sum_{i=0}^{n} C_n^i t^i (1-t)^{n-i} = [(1-t)+t]^n \equiv 1$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



Properties of Bernstein polynomial

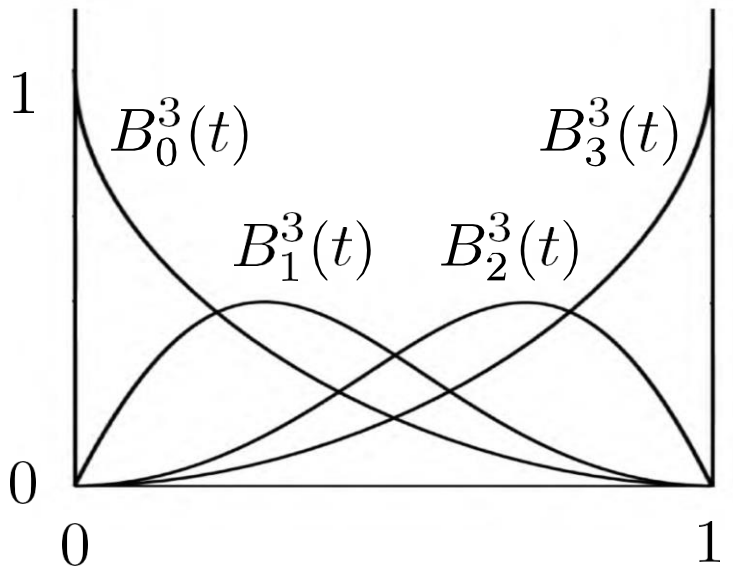
- 对称性 (Symmetry)

$$B_{i,n}(1-t) = B_{n-i,n}(t)$$

– 证明:

$$B_{n-i,n}(t) = C_n^{n-i}[1-(1-t)]^{n-(n-i)} \cdot (1-t)^{n-i}$$
$$= C_n^i t^i (1-t)^{n-i} = B_{i,n}(1-t)$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



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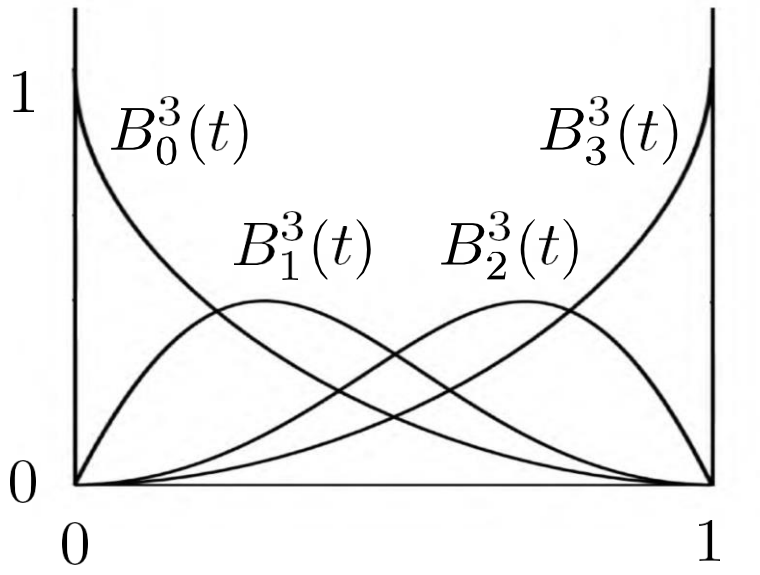


Properties of Bernstein polynomial

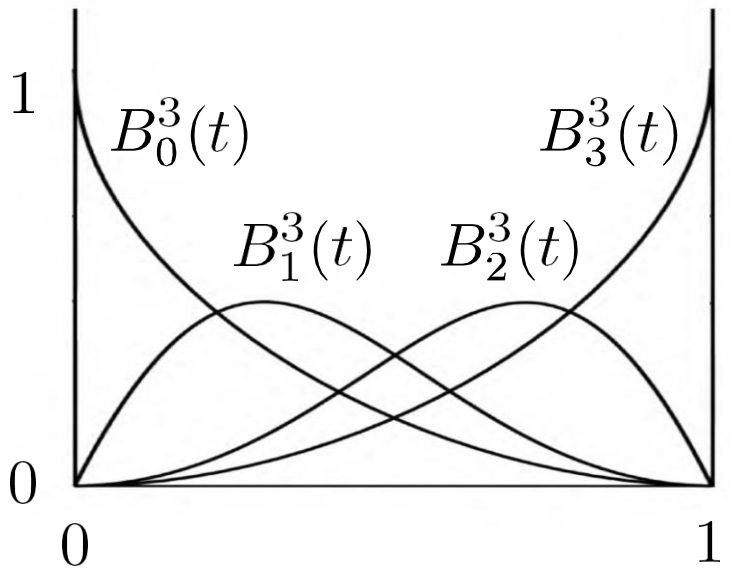
- 递归性 (Recursive)

$$B_{i,n}(t) = (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t),$$
$$(i = 0,1,\ldots,n)$$

– 这个式子表明n阶*Bernstein*多项式是n-1阶*Bernstein*多项式的线性组合。

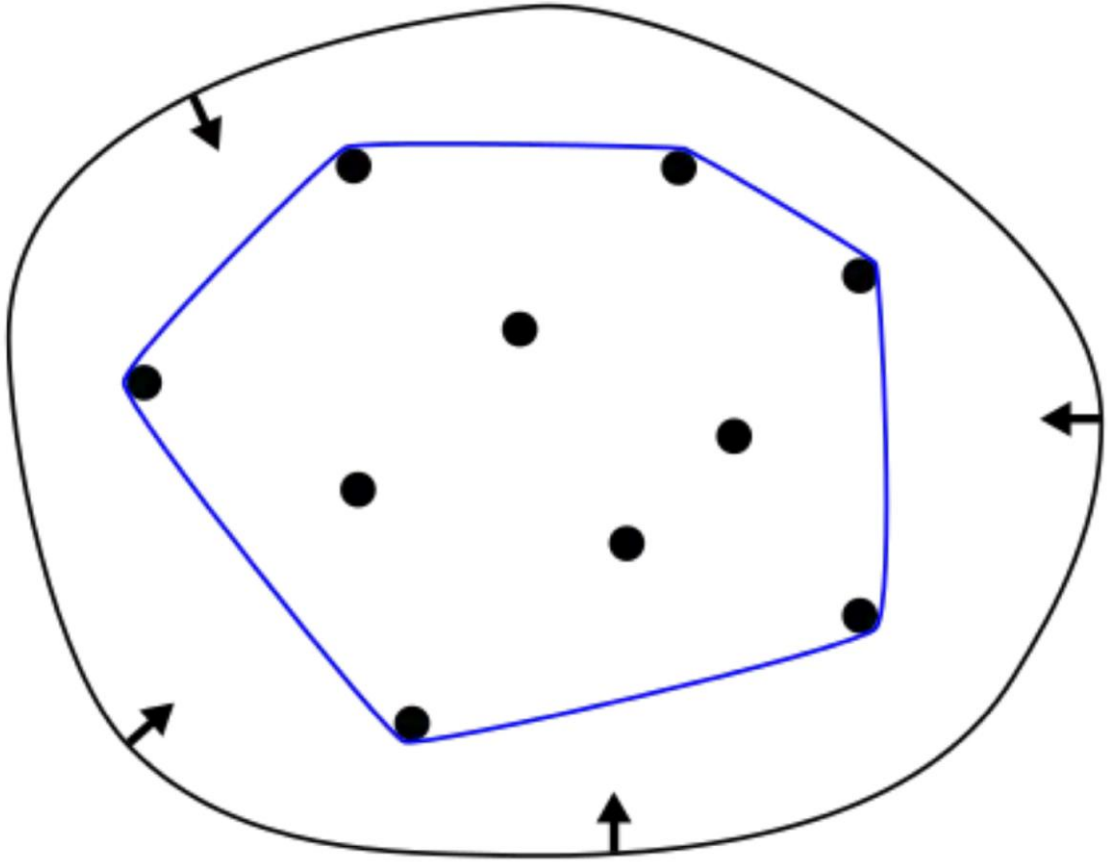
$$\begin{aligned} B_{i,n}(t) &= C_n^i t^i (1-t)^{n-i} = (C_{n-1}^i + C_{n-1}^{i-1}) t^i (1-t)^{n-i} \\ &= (1-t)C_{n-1}^i t^i (1-t)^{(n-1)-i} + tC_{n-1}^{i-1} t^{i-1} (1-t)^{(n-1)-(i-1)} \\ &= (1-t)B_{i,n-1}(t) + tB_{i-1,n-1}(t) \end{aligned}$$

$$B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$



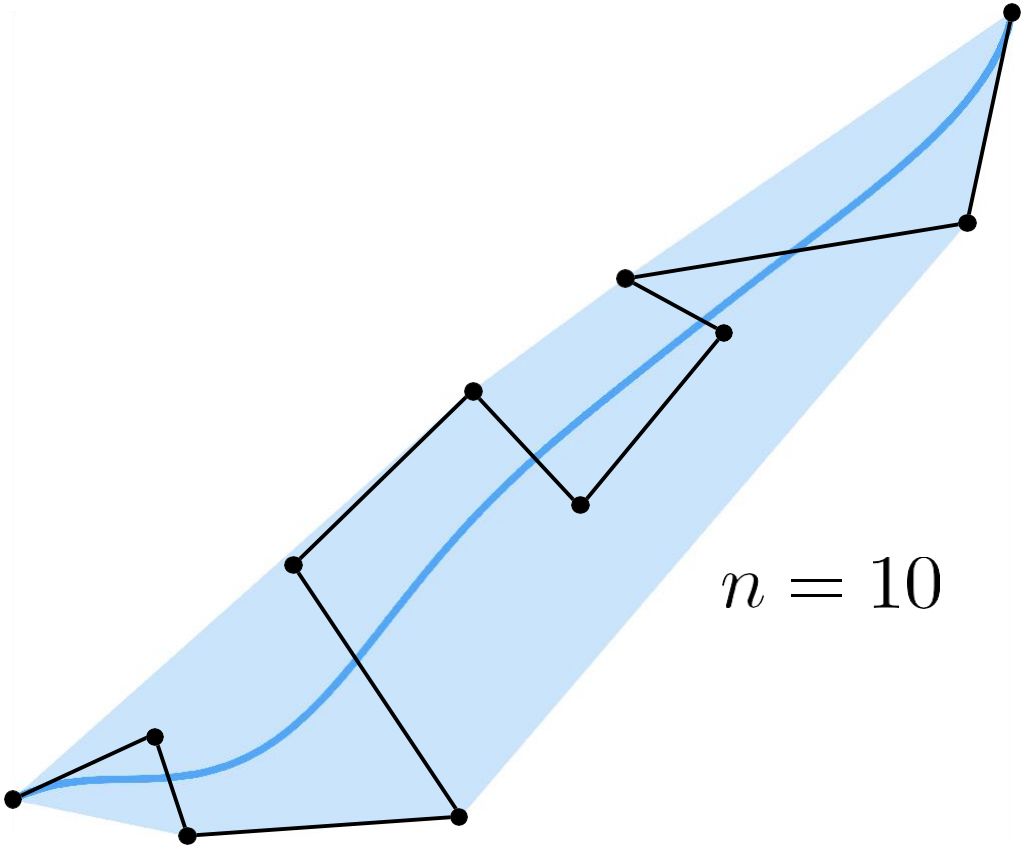
Piecewise Bézier Curves

BTW: What's a Convex Hull



[from Wikipedia]

Higher-Order Bézier Curves?



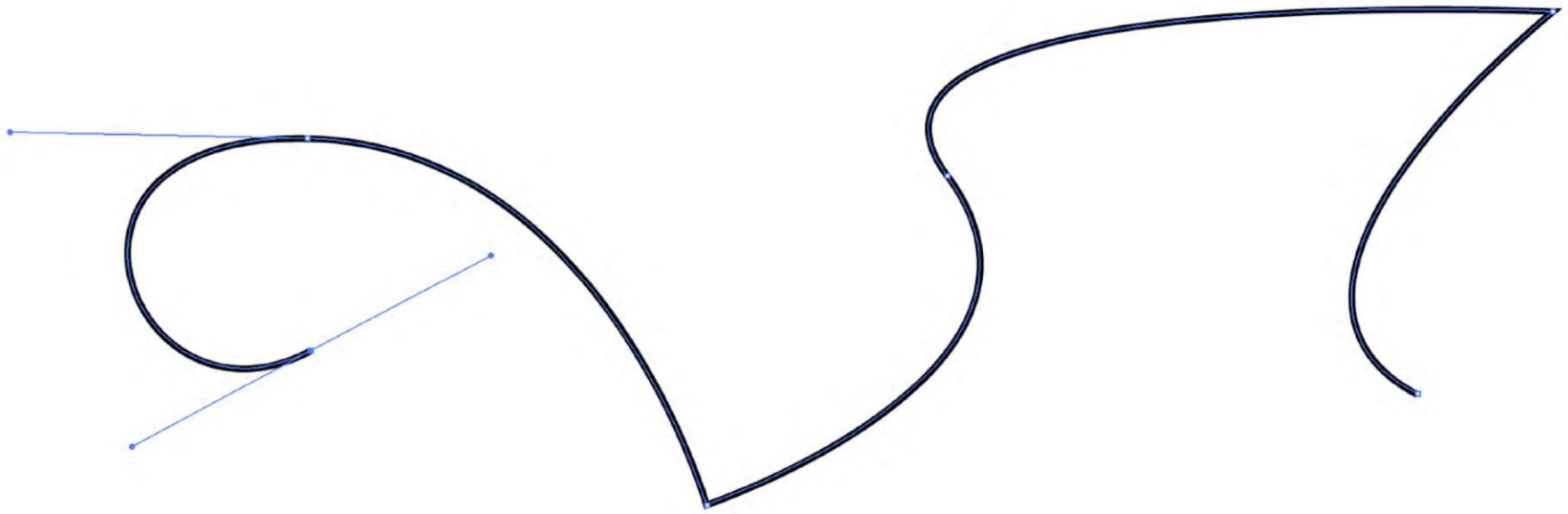
Very hard to control!

Uncommon

Piecewise Bézier Curves

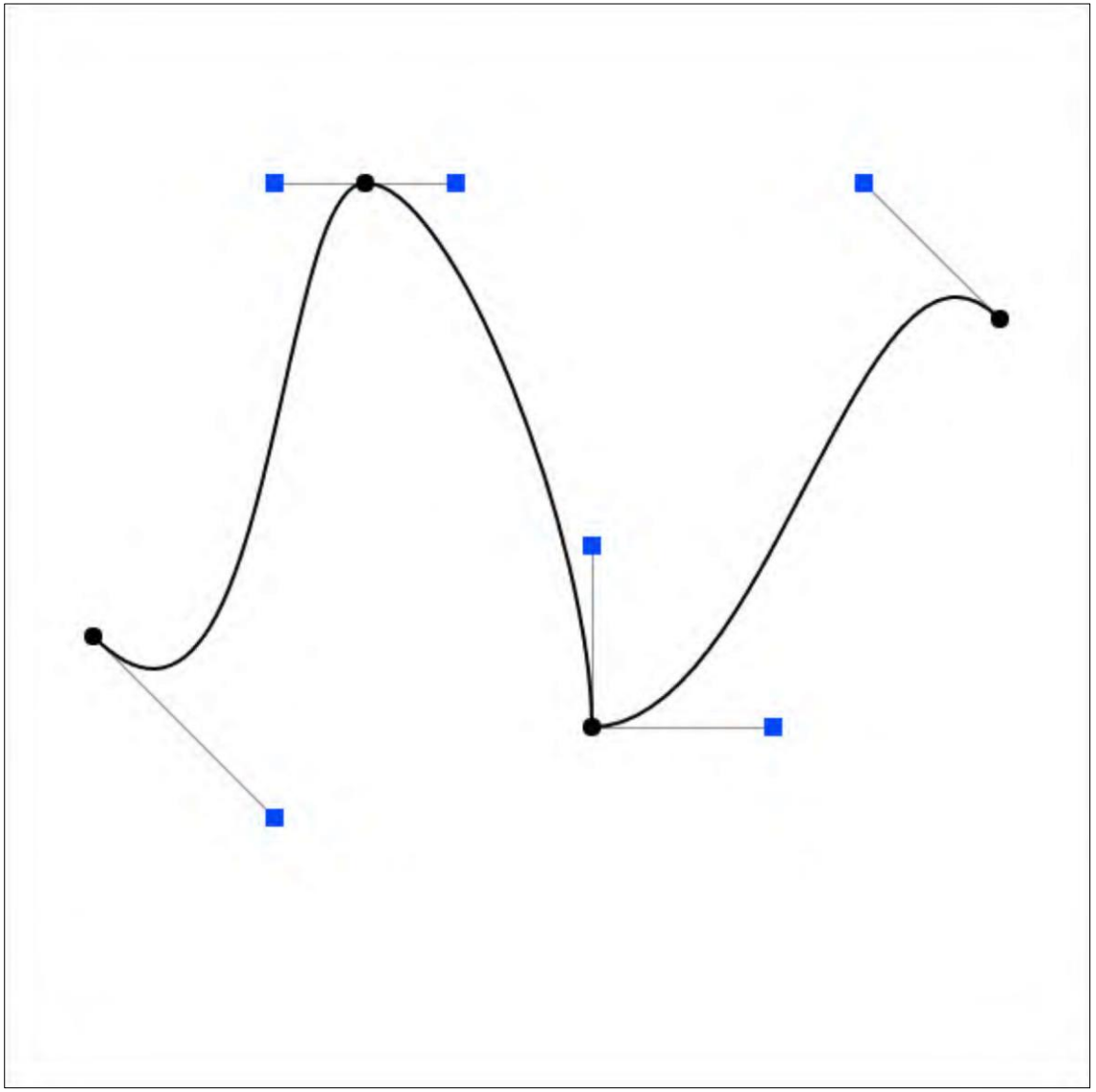
Instead, chain many low-order Bézier curve

Piecewise cubic Bézier (分段三次贝塞尔) the most common



Widely used (fonts, paths, Illustrator, Keynote, …)

Demo – Piecewise Cubic Bézier Curve



David Eck, http://math.hws.edu/eck/cs424/notes2013/canvas/bezier.html

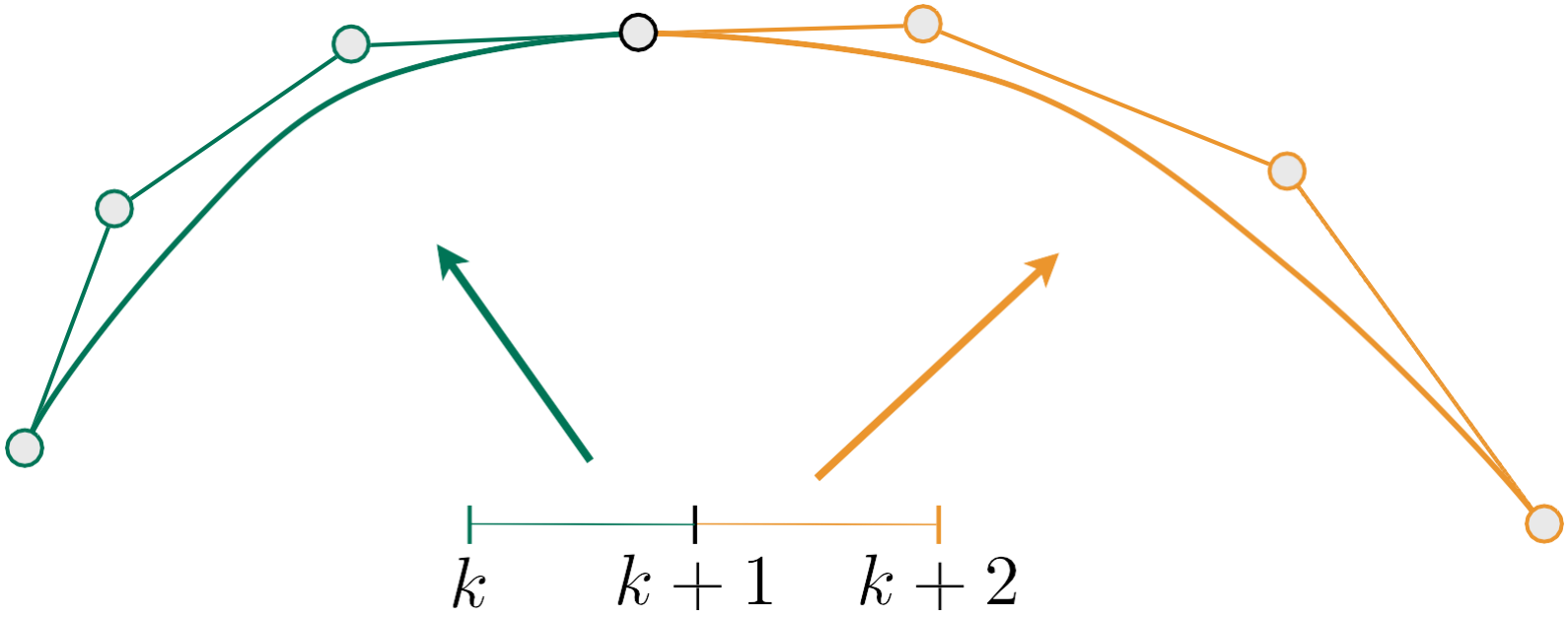
Piecewise Bézier Curve – Continuity

Two Bézier curves

$$\mathbf{a} : [k, k+1] \rightarrow \mathbb{R}^N$$

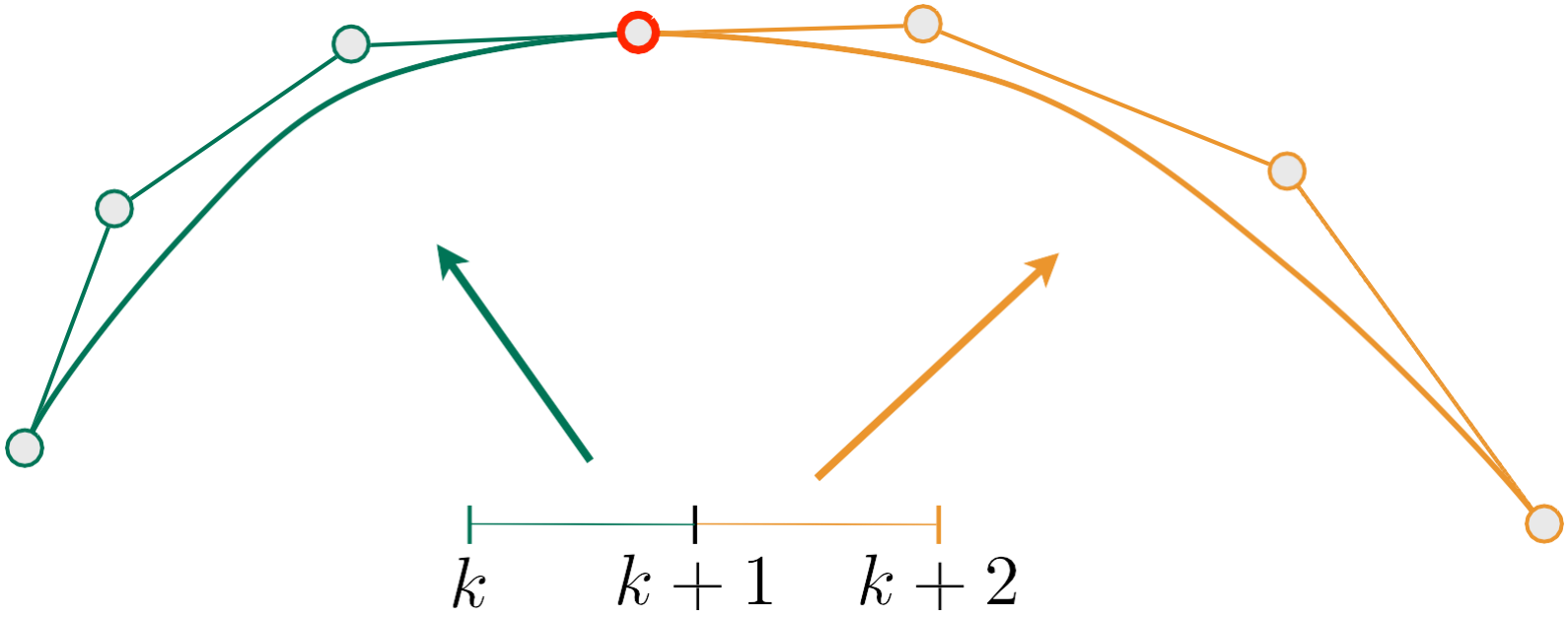
$$\mathbf{b} : [k+1, k+2] \rightarrow \mathbb{R}^N$$

Assuming integer partitions here, can generalize



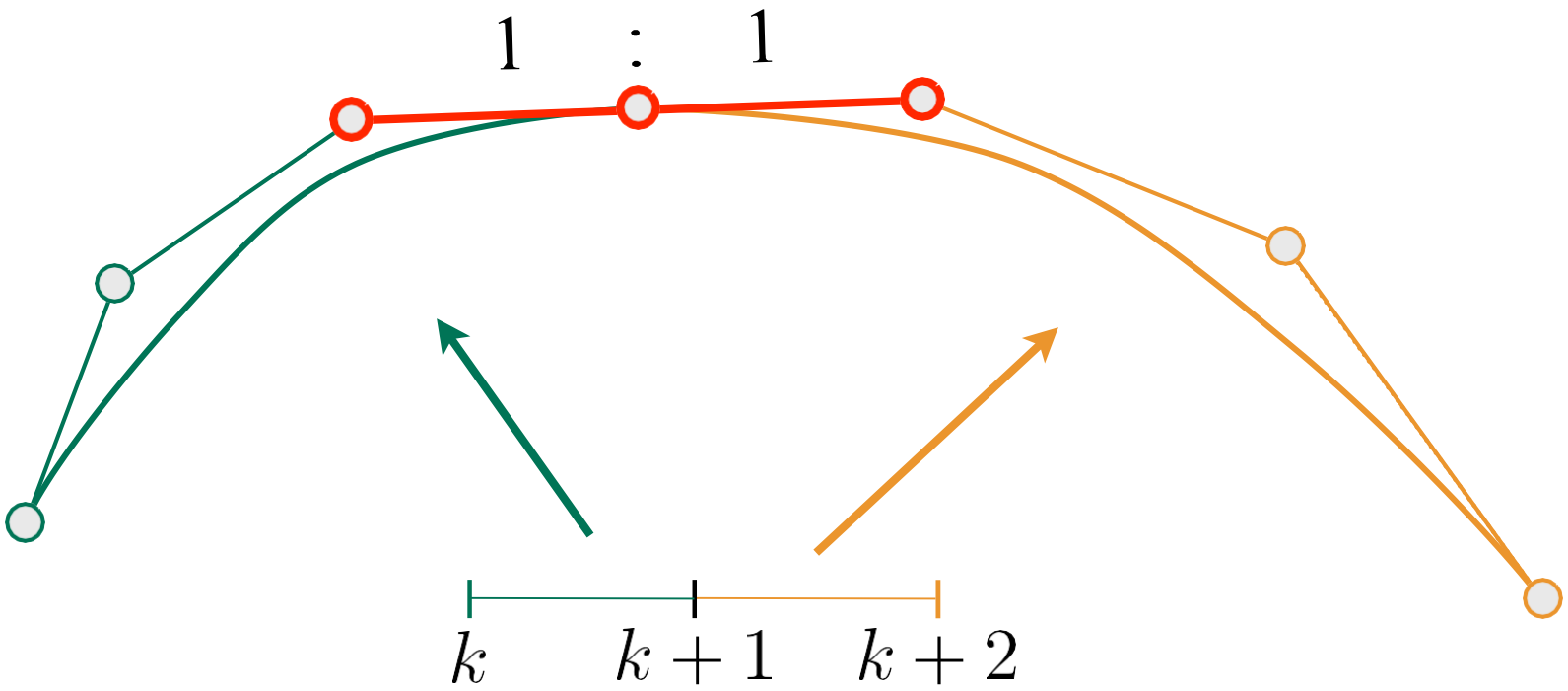
Piecewise Bézier Curve – Continuity

C^0 continuity: $\mathbf{a}_n = \mathbf{b}_0$



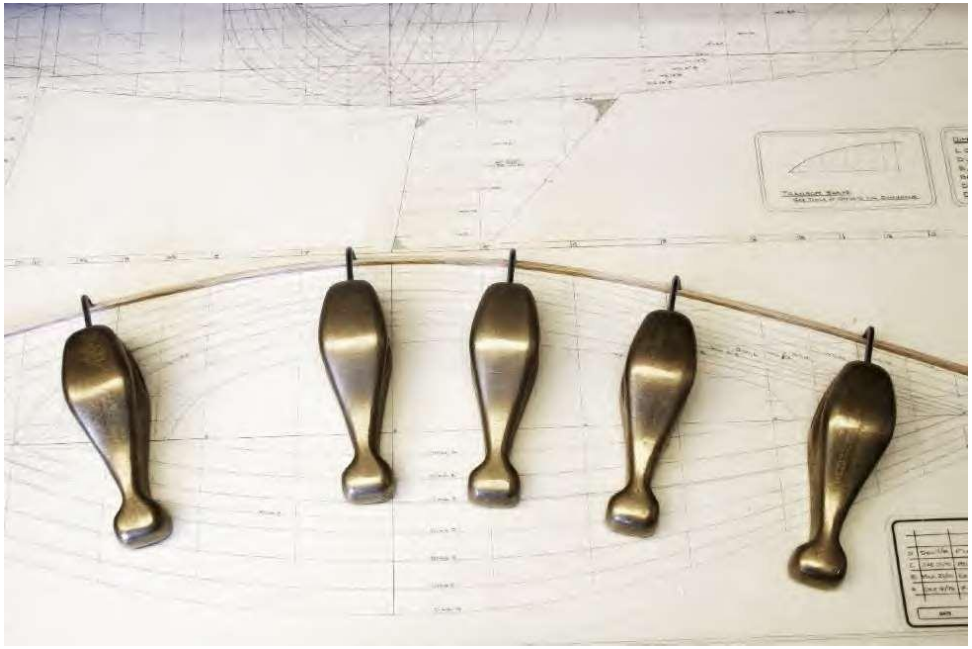
Piecewise Bézier Curve – Continuity

C^1 continuity: $$\mathbf{a}_n = \mathbf{b}_0 = \frac{1}{2}\left(\mathbf{a}_{n-1} + \mathbf{b}_1\right)$$



Other types of splines

- Spline (样条)
  - a continuous curve constructed so as to pass through a given set of points and have a certain number of continuous derivatives.
  - In short, a curve under control



A Real Draftsman's Spline

http://www.alatown.com/spline-history-architecture/

Other types of splines

- B-splines (B样条)
  - Short for basis splines
  - Require more information than Bezier curves
  - Satisfy all important properties that Bézier curves have (i.e. superset)

https://en.wikipedia.org/wiki/B-spline

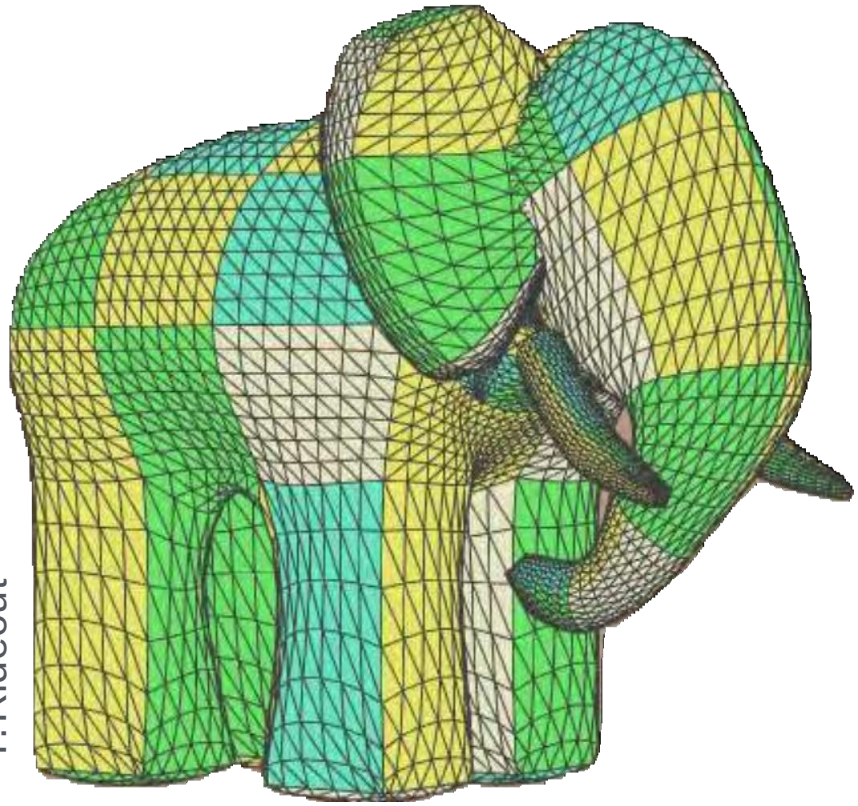
Important Note

- In this course
  - We do not cover B-splines and NURBS
  - We also do not cover operationson curves (e.g. increasing/decreasing orders, etc.)
  - To learn more / deeper, you are welcome to refer to Prof. Shi-Min Hu's course:

https://www.bilibili.com/video/BV1at411D7YV?p=13&vd_source=c9ca290d3dbf1895a1a4e9e33b4c524a

Bézier Surfaces

Extend Bézier curves to surfaces



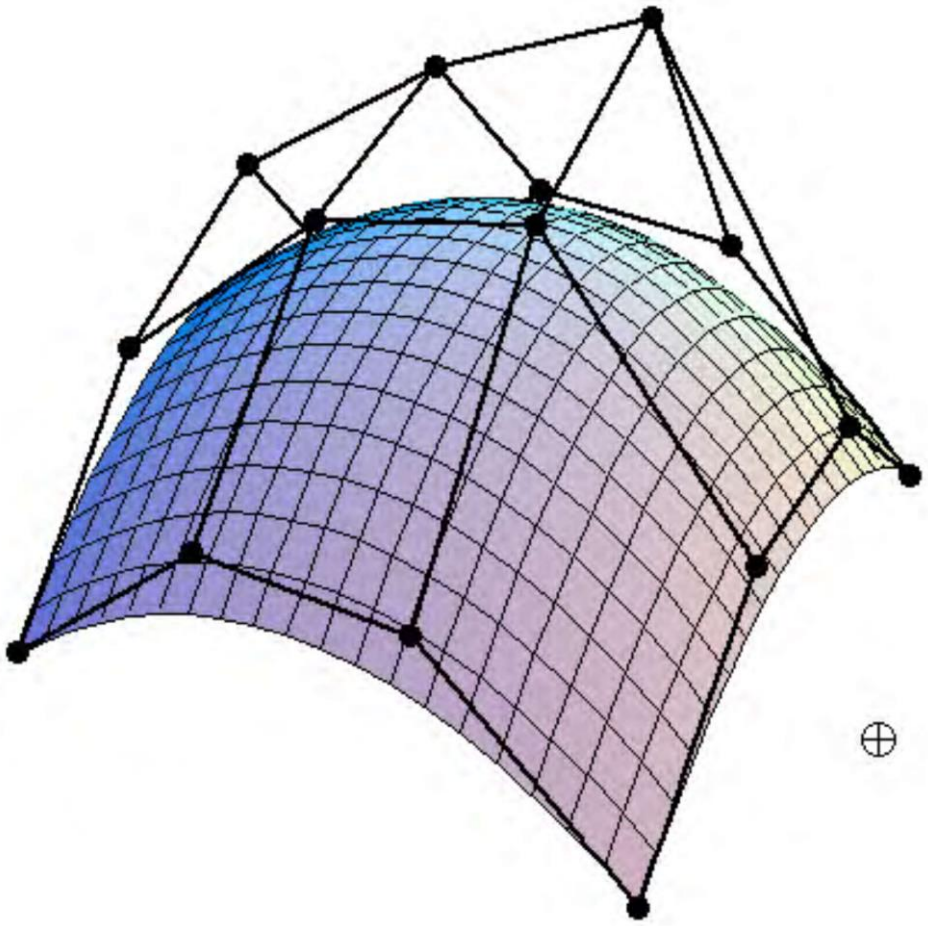
P.Rideout

Ed Catmull's "Gumbo" model

renderspirit.com

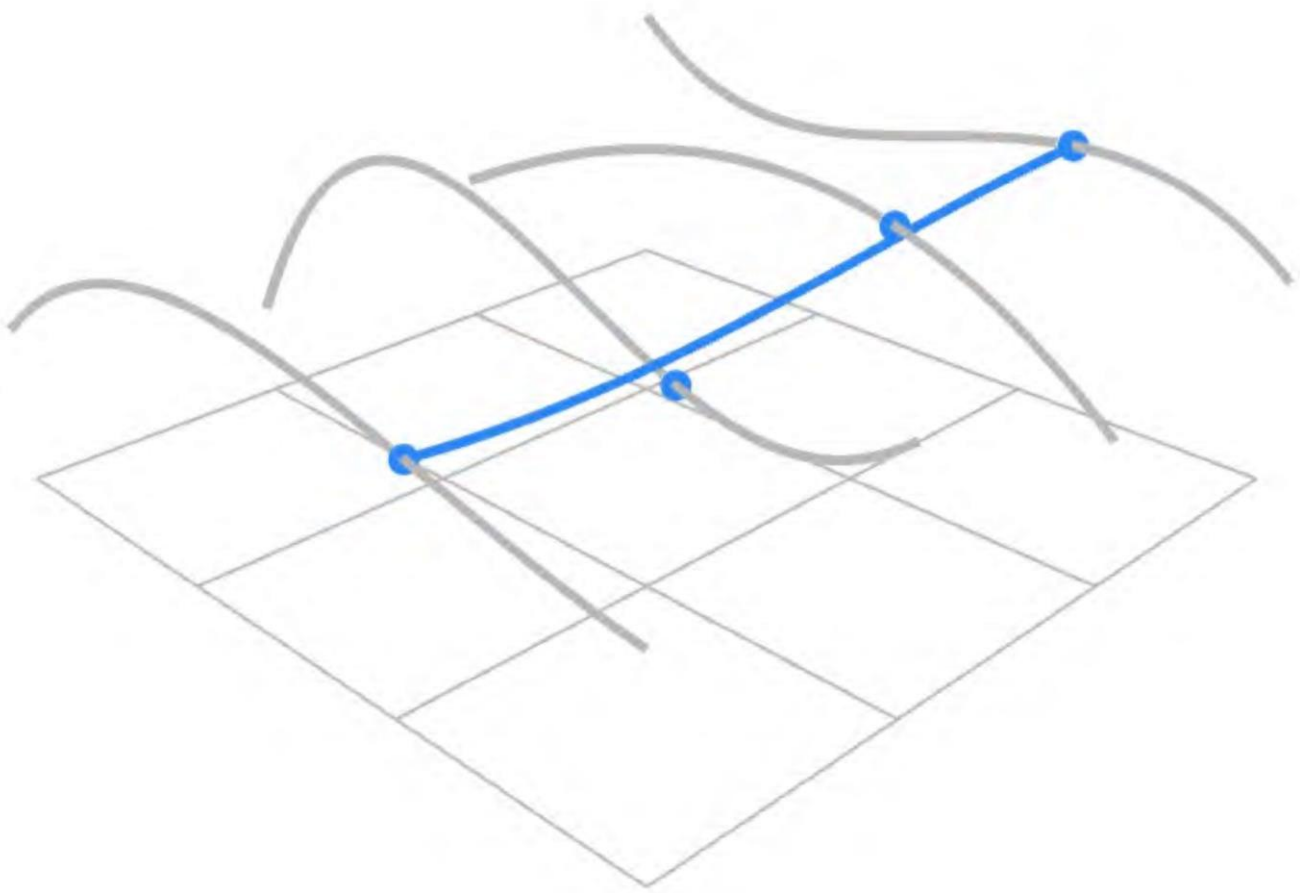
Utah Teapot

Bicubic Bézier Surface Patch



Bezier surface and 4 x 4 array of control points

Visualizing Bicubic Bézier Surface Patch



Animation: Steven Wittens, Making Things with Maths, http://acko.net

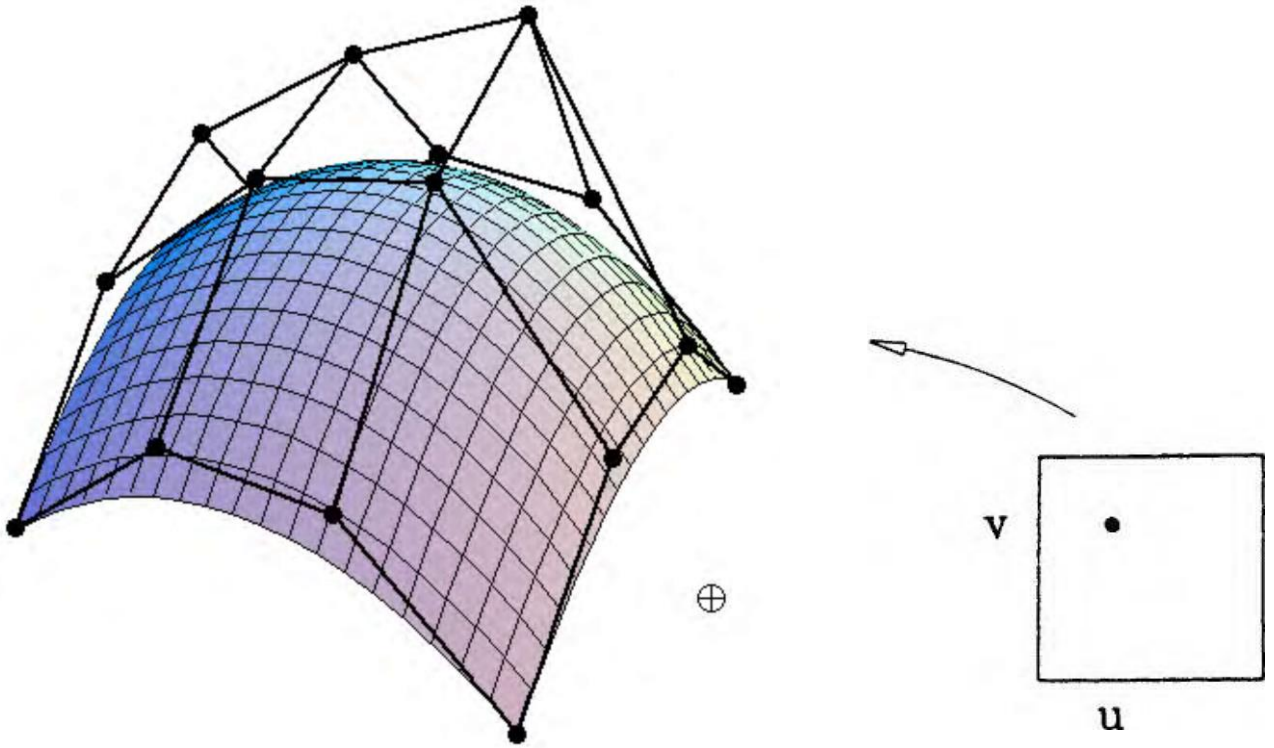
Evaluating Bézier Surfaces

Evaluating Surface Position For Parameters (u,v)

For bi-cubic Bezier surface patch,

Input: 4x4 control points

Output is 2D surface parameterized by (u,v) in $[0,1]^2$

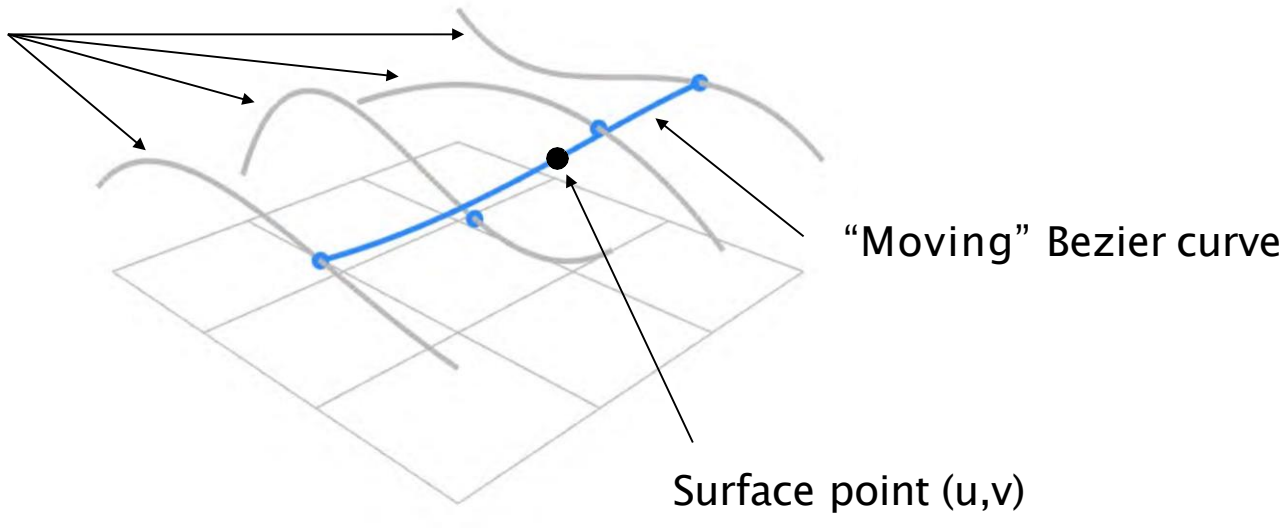


Method: Separable 1D de Casteljau Algorithm

Goal: Evaluate surface position corresponding to (u,v)

(u,v)-separable application of de Casteljau algorithm

- Use de Casteljau to evaluate point u on each of the 4 Bezier curves in u. This gives 4 control points for the "moving" Bezier curve
- Use 1D de Casteljau to evaluate point v on the "moving" curve



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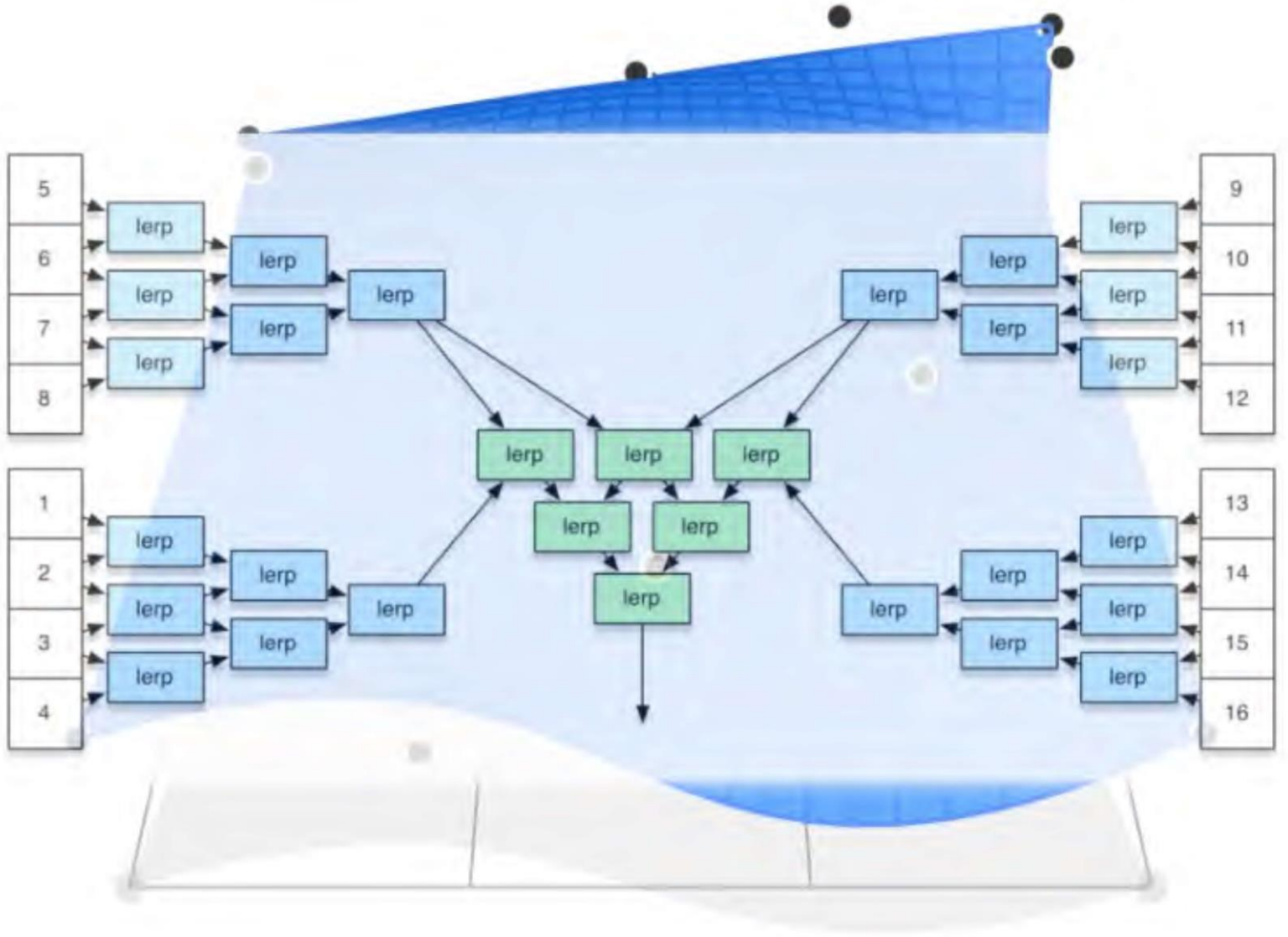

Thank you!